Riemannian Geometry on Some Noncommutative Spaces

By

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Abstract

This dissertation enquires into how the theory and mechanism of Riemannian geometry can be introduced into and integrated with the existent ones in noncommutative geometry, a branch of mathematics inspired by the development of quantum physics that concentrates on C*-algebras and related research. In conformity with the Gelfand duality, a cornerstone theorem in noncommutative geometry that establishes a one-to-one correspondence between commutative C*-algebras and locally compact Hausdorff spaces, it is suggested that a noncommutative C*-algebra notionally be deemed a "virtual noncommutative space". Based on this ideology are some forms of Riemannian geometry anticipated to reincarnate on C*-algebras.

J. Rosenberg demonstrated such a reincarnation on noncommutative tori. Especially, a corresponding adaptation of the Fundamental Theorem of Riemannian Geometry was attained. Moreover, based on this adaptation, he established a variant of the Gauß-Bonnet Theorem for noncommutative 2-tori. M. A. Peterka and A. J.-L. Sheu subsequently presented extensions and generalisations to the framework developed by Rosenberg. Specifically, an enhanced Gauß-Bonnet Theorem was substantiated for noncommutative 2-tori.

In this dissertation, we shall first tender results that are closely related to the aforementioned work on noncommutative tori, proposing several extensions of the two Gauß-Bonnet Theorems already obtained for noncommutative 2-tori and exhibiting extensions of the theorem for two special cases on noncommutative 4-tori. Thereafter, we shall transcribe Rosenberg's framework and results for quantum discs and 2-spheres with a version of the Fundamental Theorem proved. Finally, an asymptotic

behaviour of the total curvature will be demonstrated for quantum complex projective lines as an illustrative example.

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Contents

Chapter 1

Overture

1.1 Operator Algebras

To embark on our quest in the noncommutative realm, we first survey as a compass a heuristic exemplar from the commutative enclave. For lucidity and convenience, the underlying set and the underlying complex vector space of a complex algebra $\mathcal{A}$, respectively, will be denoted by $\mathrm{Set}(\mathcal{A})$ and $\mathrm{Vect}_{\mathbb{C}}(\mathcal{A})$ in the sequel.

Example 1.1.1. Let X be a locally compact Hausdorff space (pp.117, 131 of [7]). Denote by $C_0(X)$ the space of all continuous functions from X to $\mathbb{C}$ that vanishes at infinity (p.132 of [7]). Some renowned attributes ensue.

1. Equipped with the pointwise multiplication, $C_0(X)$ forms a complex algebra (p.139 of [7]).
2. Denote by $\|\cdot\|_\infty$ the uniform norm of bounded functions (p.121 of [7]). Then $(\mathrm{Vect}_{\mathbb{C}}(C_0(X)), \|\cdot\|_\infty)$ forms a complex normed vector space (p.132 of [7]).
3. $\|fg\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$ for all $f,g \in C_0(X)$.
4. $(\mathrm{Vect}_{\mathbb{C}}(C_0(X)), \|\cdot\|_\infty)$ is a complete normed vector space (Proposition 4.35 in [7]).
5. The complex conjugate forms an endomorphism on $\mathrm{Set}(C_0(X))$ that satisfies
   (a) $\overline{\overline{f}} = f$ for all $f \in C_0(X)$,
   (b) $\overline{\lambda f + g} = \overline{\lambda}\,\overline{f} + \overline{g}$ for all $f,g \in C_0(X)$ and $\lambda \in \mathbb{C}$, and
   (c) $\overline{fg} = \overline{g}\,\overline{f}$ for all $f,g \in C_0(X)$.
6. $\left\|\overline{f}f\right\|_\infty = \|f\|_\infty^2$ for all $f \in C_0(X)$.
7. $fg = gf$ for all $f,g \in C_0(X)$.
8. If X is compact, then $C_0(X)$ coincides with $C(X)$ and, thus, 1_X belongs to $C_0(X)$ and $C_0(X)$ is

unital.

9. The Riesz-Markov-Kakutani Representation Theorem for $\mathcal{B}(C_0(X),\mathbb{C})$ (Theorem 7.17 in [7]). Denote by $\mathfrak{M}(X)$ the space of all complex Radon measures on X (p.222 of [7]). Define $I : \mathfrak{M}(X) \longrightarrow \mathcal{B}(C_0(X),\mathbb{C})$ by

$$I(\mu)(\cdot) = \int_X \cdot \, d\mu \text{ for } \mu \in \mathfrak{M}(X).$$

Then I forms an isometric $\mathbb{C}$-linear isomorphism between $\mathfrak{M}(X)$ and $\mathcal{B}(C_0(X),\mathbb{C})$.

The quintessence of noncommutative geometry abides in seeking generalisations for classical geometry whilst Attributes 1 - 6 of the foregoing example are retained but Attribute 7 is partially or entirely relaxed, resembling the scene that quantum mechanics, the catalyst for the development of noncommutative geometry, manifests as a generalisation when the Uncertainty Principle is incorporated into Newtonian mechanics. Thus, as our first stride beneath the noncommutative welkin, we consider extending Attributes 1 - 6 to arbitrary complex algebras. Intrigued readers could refer to [15] or [14] for more details.

Definition 1.1.1. Let $\mathcal{A}$ be a complex algebra and $\|\cdot\|$ a norm on $\mathrm{Vect}_{\mathbb{C}}(\mathcal{A})$. Then $\|\cdot\|$ is called a norm on $\mathcal{A}$ if $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a,b \in \mathcal{A}$. In this case, $(\mathcal{A},\|\cdot\|)$ is called a normed algebra.

Remark. This regains Attributes 2 and 3 on complex algebras.

Definition 1.1.2. Let $(\mathcal{A},\|\cdot\|)$ be a normed algebra. Then $(\mathcal{A},\|\cdot\|)$ is called a Banach algebra if $(\mathrm{Vect}_{\mathbb{C}}(\mathcal{A}),\|\cdot\|)$ is a complete normed vector space.

Remark. This regains Attribute 4 on normed algebras.

Definition 1.1.3. Let $\mathcal{A}$ be a complex algebra and * an endomorphism on $\mathrm{Set}(\mathcal{A})$. Then $(\mathcal{A},^*)$ is called a *-algebra if

(a) $(a^*)^* = a$ for all $a \in \mathcal{A}$,

(b) $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$ for all $a,b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, and

(c) $(ab)^* = b^* a^*$ for all $a,b \in \mathcal{A}$.

Remark.

1. This regains Attribute 5 on complex algebras.
2. If $\|\cdot\|$ is a norm on $\mathcal{A}$, then $(\mathcal{A},^*, \|\cdot\|)$ will be called a normed *-algebra. Thus, if $(\mathcal{A}, \|\cdot\|)$ is a Banach algebra, then $(\mathcal{A},^*, \|\cdot\|)$ shall be called a Banach *-algebra.
3. If $\mathcal{A}$ is unital, then $\mathbb{1}_{\mathcal{A}}^* = \mathbb{1}_{\mathcal{A}}$.

Definition 1.1.4. Let $(\mathcal{A},^*, \|\cdot\|)$ be a normed *-algebra. Then $\|\cdot\|$ is called a C*-norm on $(\mathcal{A},^*)$ if $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

Remark.

1. This regains Attribute 6 on normed *-algebras.
2. $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.
3. If $\mathcal{A}$ is unital, then $\|\mathbb{1}_{\mathcal{A}}\| = 1$.

Definition 1.1.5. Let $\mathcal{A}$ be a *-algebra and $\|\cdot\|$ a C*-norm on $\mathcal{A}$. Then $(\mathcal{A}, \|\cdot\|)$ is called a C*-algebra if it is a Banach *-algebra.

Remark. This regains Attributes 1 - 6 on complex algebras and will be the prime object for our study.

Example 1.1.2. Satisfying all of Attributes 1 - 6, the prototypical $(C_0(X), \overline{\cdot}, \|\cdot\|_\infty)$ is naturally a C*-algebra. Specifically, it is a commutative one.

Example 1.1.3. Let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the space of all bounded $\mathbb{C}$-linear operators on $\mathcal{H}$. Some pertinent characteristics ensue.

1. Equipped with the composition of maps, $\mathcal{B}(\mathcal{H})$ forms a complex algebra (Theorem 15.8 in [12]).
2. Denote by $\|\cdot\|_{\text{op}}$ the operator norm. Then $\big(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\text{op}}\big)$ forms a Banach algebra (Theorems 15.3, 15.8 in [12]).
3. Denote by * the Hermitian conjugate. Then $(\mathcal{B}(\mathcal{H}),^*)$ forms a *-algebra (p.222 of [12]).
4. $\|\cdot\|_{\text{op}}$ is a C*-norm on $(\mathcal{B}(\mathcal{H}),^*)$ (Theorem 19.14 in [12]).

Thus, $\left(\mathcal{B}(\mathcal{H}),^*,\|\cdot\|_{\mathrm{op}}\right)$ forms a C*-algebra. Specifically, it is a noncommutative one if $\dim(\mathcal{H}) > 1$. Moreover, $\mathcal{B}(\mathcal{H})$ reductively coincides with $\mathrm{M}_n(\mathbb{C})$ if $\dim(\mathcal{H}) = n$.

After formulating the mathematical objects that will be investigated, we anticipate that they possess internal structures.

Definition 1.1.6. Let $(\mathcal{A},^*,\|\cdot\|)$ be a C*-algebra and $\mathcal{B}$ a subalgebra of $\mathcal{A}$.

1. $(\mathcal{B},^*)$ is called a *-subalgebra of $(\mathcal{A},^*)$ if it is a *-algebra.
2. $(\mathcal{B},^*,\|\cdot\|)$ is called a C*-subalgebra of $(\mathcal{A},^*,\|\cdot\|)$ if it is a C*-algebra.

Remark. If $(\mathcal{B},^*)$ is a *-subalgebra of $(\mathcal{A},^*)$, then $(\mathcal{B},^*,\|\cdot\|)$ will be called a normed *-subalgebra of $(\mathcal{A},^*,\|\cdot\|)$.

Theorem 1.1.1 (Corollary 1.5.3 in [15])**.** Every closed ideal of a C*-algebra is necessarily closed under * and, thus, forms a C*-subalgebra.

Example 1.1.4. Let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathcal{K}(\mathcal{H})$ the space of all compact $\mathbb{C}$-linear operators on $\mathcal{H}$ (p.233 of [12]). It has been widely known that $\left(\mathcal{K}(\mathcal{H}),\|\cdot\|_{\mathrm{op}}\right)$ forms a closed ideal of $\mathcal{B}(\mathcal{H})$ (Theorem 21.1 in [12]). Thus, $\left(\mathcal{K}(\mathcal{H}),^*,\|\cdot\|_{\mathrm{op}}\right)$ forms a C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Naturally, it is a noncommutative one if $\dim(\mathcal{H}) > 1$. Moreover, it is a nonunital one if $\dim(\mathcal{H})$ is infinite.

Corollary 1.1.1 (Corollary 1.5.5 in [15])**.** Let $\mathcal{A}$ be a C*-algebra and $\mathcal{I}$ a closed ideal. Then, equipped with the quotient norm, $\mathcal{A}/\mathcal{I}$ forms a C*-algebra.

Example 1.1.5. Pursuant to Example 1.1.4, $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ forms a C*-algebra.

Besides internal structures, it is expected that there exist maps that preserve these structures.

Definition 1.1.7. Let f be a map between *-algebras. Then f is called

1. a *-algebra homomorphism if
   (a) it is a complex algebra homomorphism and
   (b) $f(a^*) = f(a)^*$ for all $a \in \mathrm{Dom}(f)$, and

2. a *-algebra isomorphism if it is a bijective *-algebra homomorphism.

Definition 1.1.8. Let f be a map between C*-algebras. Then f is called

1. a C*-algebra homomorphism if it is a continuous *-algebra homomorphism, and
2. a C*-algebra isomorphism if it is an isometric *-algebra isomorphism.

Example 1.1.6. Let X, Y be locally compact Hausdorff spaces and ϕ a continuous map from X to Y. Denote by ϕ^* the pullback of ϕ from $C_0(Y)$ to $C_0(X)$, videlicet,

$$\phi^*(f) = f \circ \phi \text{ for all } f \in C_0(Y).$$

Some pertinent characteristics ensue.

1. ϕ^* is $\mathbb{C}$-linear.
2. $\phi^*(fg) = \phi^*(f)\phi^*(g)$ for all $f, g \in C_0(Y)$.
3. $\phi^*\left(\overline{f}\right) = \overline{\phi^*(f)}$ for all $f \in C_0(Y)$.
4. $\|\phi^*(f)\|_\infty \leq \|f\|_\infty$ for all $f \in C_0(Y)$.

Thus, ϕ^* forms a C*-algebra homomorphism. Especially, if ϕ is surjective, then ϕ^* is isometric. Moreover, if ϕ is a homeomorphism, then ϕ^* is a C*-algebra isomorphism.

Example 1.1.7.

1. Pursuant to Example 1.1.4, the canonical embedding of $\mathcal{K}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$ is naturally a C*-algebra homomorphism. Specifically, it is an isometric one.
2. Pursuant to Example 1.1.5, the canonical projection of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is naturally a C*-algebra homomorphism. Specifically, it is a norm-decreasing one.

Theorem 1.1.2 (Theorem 1.5.7 in [15])**.** Every *-algebra homomorphism between C*-algebras is necessarily norm-decreasing and, thus, is a continuous map. Moreover, if such a map is injective, then it is necessarily isometric.

The three renowned isomorphism theorems still hold for C*-algebras.

Theorem 1.1.3 (First Isomorphism Theorem, Theorem 1.5.7 in [15])**.** Let f be a *-algebra homomorphism between C*-algebras. Then

(a) $\mathrm{Ker}(f)$ forms a closed ideal of $\mathrm{Dom}(f)$,

(b) $\mathrm{Im}(f)$ forms a C*-subalgebra of $\mathrm{Codom}(f)$, and

(c) $\mathrm{Dom}(f)/\mathrm{Ker}(f)$ is isomorphic to $\mathrm{Im}(f)$.

Theorem 1.1.4 (Second Isomorphism Theorem, Corollary 1.5.8 in [15])**.** Let $\mathcal{A}$ be a C*-algebra, and $\mathcal{B}$ a C*-subalgebra of $\mathcal{A}$ and $\mathcal{I}$ a closed ideal of $\mathcal{A}$. Then

(a) $\mathcal{B}+\mathcal{I}$ forms a C*-subalgebra of $\mathcal{A}$,

(b) $\mathcal{B}\cap\mathcal{I}$ forms a closed ideal of $\mathcal{B}$, and

(c) $(\mathcal{B}+\mathcal{I})/\mathcal{I}$ is isomorphic to $\mathcal{B}/(\mathcal{B}\cap\mathcal{I})$.

Theorem 1.1.5 (Third Isomorphism Theorem)**.** Let $\mathcal{A}$ be a C*-algebra and $\mathcal{I}$ a closed ideal.

1. Let $\mathcal{B}'$ be a subset of $\mathcal{A}/\mathcal{I}$. Then $\mathcal{B}'$ forms a C*-subalgebra of $\mathcal{A}/\mathcal{I}$ if and only if there exists a C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing $\mathcal{I}$ such that $\mathcal{B}' = \mathcal{B}/\mathcal{I}$.
2. Let $\mathcal{J}'$ be a subset of $\mathcal{A}/\mathcal{I}$. Then $\mathcal{J}'$ forms a closed ideal of $\mathcal{A}/\mathcal{I}$ if and only if there exists a closed ideal $\mathcal{J}$ of $\mathcal{A}$ containing $\mathcal{I}$ such that $\mathcal{J}' = \mathcal{J}/\mathcal{I}$.
3. Let $\mathcal{J}$ be a closed ideal of $\mathcal{A}$ with $\mathcal{I} \subseteq \mathcal{J}$. Then $(\mathcal{A}/\mathcal{I})/(\mathcal{J}/\mathcal{I})$ is isomorphic to $\mathcal{A}/\mathcal{J}$.

Proof. It suffices to validate the biconditional statement in 1 that $\mathcal{B}/\mathcal{I}$ is closed if and only if $\mathcal{B}$ is closed.

"$\Rightarrow$"

Denote by ϕ the canonical projection of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$. Then $\mathcal{B} = \phi^{-1}(\mathcal{B}/\mathcal{I})$ and, thus, $\mathcal{B}$ is closed.

"$\Leftarrow$"

Denote by ψ the canonical embedding of $\mathcal{B}/\mathcal{I}$ into $\mathcal{A}/\mathcal{I}$. Then ψ is isometric and, thus, $\mathcal{B}/\mathcal{I}$ is closed. □

There is a crucial class in the aforementioned maps that merits an exclusive appellation for prospective utilisation.

Definition 1.1.9. Let ρ be a *-algebra homomorphism. If $\mathrm{Codom}(\rho) = \mathcal{B}(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$, then ρ will be called a (*-algebra) representation of $\mathrm{Dom}(\rho)$ on $\mathcal{H}$ and $\mathrm{Dom}(\rho)$ is said to be (*-algebraically) represented on $\mathcal{H}$ by ρ.

Remark. If $\mathrm{Dom}(\rho)$ is a C*-algebra, then ρ is continuous.

Definition 1.1.10. Let $\mathcal{A}$ be a *-algebra and ρ a representation of $\mathcal{A}$. Then ρ is said to be faithful if it is injective. In this case, $\mathcal{A}$ is said to be faithfully represented by ρ.

Remark. If $\mathcal{A}$ is a C*-algebra, then ρ is isometric (when it is faithful).

Theorem 1.1.6 (Corollary 3.7.5 in [15])**.** Every C*-algebra can be faithfully represented on a complex Hilbert space.

Remark. This evinces why this section is entitled "Operator Algebras".

Example 1.1.8. Let X be a locally compact Hausdorff space and μ a positive element in $\mathfrak{M}(X)$ (Attribute 9 in Example 1.1.1). Denote by $\mathfrak{B}_X$ the Borel σ-algebra on X (p.22 of [7]) and $\|\cdot\|_2$ the L^2-norm (p.181 of [7]). Moreover, denote by $L^2(X,\mathfrak{B}_X,\mu)$ the space of all measurable functions from X to $\mathbb{C}$ with finite L^2-norms (pp.43, 181 of [7]). It has been widely known that $L^2(X,\mathfrak{B}_X,\mu)$ forms a complex Hilbert space (p.186 of [7]). A faithful representation of $C_0(X)$ on $L^2(X,\mathfrak{B}_X,\mu)$ is explicitly constructed underneath.

Pursuant to Hölder's Inequality (Theorem 6.8 in [7]), $\|f\phi\|_2 \leq \|f\|_\infty \cdot \|\phi\|_2$ for all $\phi \in L^2(X,\mathfrak{B}_X,\mu)$ and $f \in C_0(X)$. Thus, for every $f \in C_0(X)$, the left multiplication by f, denoted by L_f, forms a $\mathbb{C}$-linear operator on $L^2(X,\mathfrak{B}_X,\mu)$ bounded by $\|f\|_\infty$. Some pertinent characteristics of the left multiplication ensue.

1. $L_{(\cdot)}$ is $\mathbb{C}$-linear.
2. $L_{fg} = L_f L_g$ for all $f,g \in C_0(X)$.
3. $\displaystyle\int_X (f\phi)\cdot\overline{\psi} = \int_X \phi \cdot \overline{\overline{f}\psi}$ for all $\phi,\psi \in L^2(X,\mathfrak{B}_X,\mu)$ and $f \in C_0(X)$.
4. $L_f = \mathbb{0}_{L^2(X,\mathfrak{B}_X,\mu)}$ if and only if $f = 0_X$.

Thence, define $\rho : C_0(X) \longrightarrow \mathcal{B}(L^2(X,\mathfrak{B}_X,\mu))$ by

$$\rho(f) = L_f \text{ for } f \in C_0(X).$$

Then ρ forms, as promised, a faithful representation of $C_0(X)$ on $L^2(X,\mathfrak{B}_X,\mu)$.

Having instituted the principal objects for research and the germane maps therebetween, we now adopt a shift from a holistic perspective on these structures to an atomistic one, concentrating on analysing and classifying elements and their relationships within these structures.

Let $\mathcal{A}$ be a complex algebra. Extend the multiplication on $\mathcal{A}$ to $\mathrm{Vect}_{\mathbb{C}}(\mathcal{A}) \oplus \mathbb{C}$ via

$$(a,z) \cdot (b,w) = (ab + zb + wa, zw),\ (a,z),(b,w) \in \mathrm{Vect}_{\mathbb{C}}(\mathcal{A}) \oplus \mathbb{C}.$$

Then $(\mathrm{Vect}_{\mathbb{C}}(\mathcal{A}) \oplus \mathbb{C}, \cdot)$ forms a complex algebra. Specifically, it is a unital one with $(0_{\mathcal{A}}, 1)$ acting as the multiplicative identity.

Definition 1.1.11. Let $\mathcal{A}$ be a complex algebra. Then $(\mathrm{Vect}_{\mathbb{C}}(\mathcal{A}) \oplus \mathbb{C}, \cdot)$ is called the unitization of $\mathcal{A}$ and will be denoted by $\mathcal{A}^+$.

Remark.

1. $\mathcal{A}$ forms an ideal of $\mathcal{A}^+$.
2. If $\mathcal{A}$ is a *-algebra, then * can be extended to $\mathcal{A}^+$ via

$$(a,z)^* = (a^*, \bar{z}) \text{ for } (a,z) \in \mathcal{A}^+$$

   and, thus, $(\mathcal{A}^+, ^*)$ forms a *-algebra.

Example 1.1.9. Let X be a locally compact Hausdorff space that is noncompact. Denote by X^+ the Alexandrov compactification of X (p.132 of [7]). Define $\phi : C_0(X)^+ \longrightarrow C_0(X^+)$ by

$$\phi(f,z) = f + z1_{X^+} \text{ for } (f,z) \in C_0(X)^+.$$

Then ϕ forms a *-algebra isomorphism.

Definition 1.1.12. Let $\mathcal{A}$ be a complex algebra and a an element of $\mathcal{A}$. Then

1. $\{z \in \mathbb{C} | a - z\mathbb{1}_{\mathcal{A}} \text{ is noninvertible in } \mathcal{A}\}$ is called the spectrum of a if $\mathcal{A}$ is unital, and
2. $\{z \in \mathbb{C} | a - z\mathbb{1}_{\mathcal{A}^+} \text{ is noninvertible in } \mathcal{A}^+\}$ is called the spectrum of a if $\mathcal{A}$ is nonunital.

In either case, the spectrum of a will be denoted by $\sigma_{\mathcal{A}}(a)$.

Remark. If $\mathcal{A}$ is a *-algebra, then $\sigma_{\mathcal{A}}(a^*) = \overline{\sigma_{\mathcal{A}}(a)}$.

Example 1.1.10.

1. Pursuant to Attribute 8 in Example 1.1.1, if X is compact, then $\sigma_{C_0(X)}(f) = \mathrm{Im}(f)$ for all $f \in C_0(X)$, whereas, by virtue of the *-algebra isomorphism in Example 1.1.9, if X is noncompact, then $\sigma_{C_0(X)}(f) = \mathrm{Im}(f) \cup \{0\}$ for all $f \in C_0(X)$.
2. Let A be an element of $\mathrm{M}_n(\mathbb{C})$. Then $\sigma_{\mathrm{M}_n(\mathbb{C})}(A)$ is, by definition, the set of all eigenvalues of A.

Theorem 1.1.7 (Theorem 17.4 in [12])**.** Let $\mathcal{A}$ be a normed algebra and a an element of $\mathcal{A}$. If $\mathcal{A}$ is unital and Banach, then

(a) $\sigma_{\mathcal{A}}(a)$ is nonempty and compact, and

(b) $\sup\{|z| : z \in \sigma_{\mathcal{A}}(a)\} = \lim\limits_{k\to\infty} \left\|a^k\right\|^{\frac{1}{k}}$.

Definition 1.1.13. Let $\mathcal{A}$ be a C*-algebra and a an element of $\mathcal{A}$. Then a is said to be

1. normal if $a^*a = aa^*$,
2. self-adjoint if $a^* = a$, and
3. positive if a is normal and $\sigma_{\mathcal{A}}(a) \subseteq [0,\infty)$.

Moreover, if $\mathcal{A}$ is unital, then a is said to be

4. unitary if $a^*a = aa^* = \mathbb{1}_{\mathcal{A}}$.

Remark.

1. If a is self-adjoint or unitary, then it is normal.
2. If a is unitary, then $\|a\| = 1$.
3. The subset of all self-adjoint elements of $\mathcal{A}$ will be denoted by $\mathcal{A}_{\mathrm{sa}}$.
4. The subset of all positive elements of $\mathcal{A}$ will be denoted by $\mathcal{A}_{\geq}$.

Proposition 1.1.1 (1.1.4 in [15])**.** Let $\mathcal{A}$ be a C*-algebra and a an element of $\mathcal{A}$. If a is normal, then $\sup\{|z| : z \in \sigma_{\mathcal{A}}(a)\}$ coincides with $\|a\|$.

Proposition 1.1.2 (Lemma 1.1.5 in [15])**.**

1. The spectrum of any unitary element of a C*-algebra is contained in $\mathbb{T}$.
2. The spectrum of any self-adjoint element of a C*-algebra is contained in $\mathbb{R}$.

1.2 Riemannian Geometry

Since the mechanism of Riemannian geometry will be attempted to transport onto C*-algebras afterwards, we temporarily digress from algebra, relishing a ramble in the entrancing garden of differential geometry. Intrigued readers could refer to [13] and [10] for more details.

Definition 1.2.1. Let M be a Hausdorff space. Then M is called an n-dimensional topological manifold if

(a) it is second countable (p.3 of [13]), and

(b) for every point p, there exists a homeomorphism x_p such that $\mathrm{Dom}(x_p)$ is an open neighborhood of p and $\mathrm{Im}(x_p)$ is an open subset of $\mathbb{R}^n$.

In this case, x_p is called a (coordinate) chart on M containing p.

Remark.

1. M is necessarily locally compact and paracompact (Theorem 1.15 in [13]).
2. Naturally, $\mathbb{R}^n$ forms an n-dimensional topological manifold, videlicet, $x_p = \mathrm{Id}_{\mathbb{R}^n}$ for all $p \in \mathbb{R}^n$.

Definition 1.2.2. Let M be a topological manifold and $\{x_\alpha\}_{\alpha\in\mathrm{A}}$ a collection of charts. Then $\{x_\alpha\}_{\alpha\in\mathrm{A}}$ is called an atlas for M if $\{\mathrm{Dom}(x_\alpha)\}_{\alpha\in\mathrm{A}}$ is a covering of M (p.601 of [13]).

Example 1.2.1 (p.3 of [10])**.** Denote by $\mathbb{S}^n$ the n-sphere, videlicet,

$$\mathbb{S}^n = \left\{ (x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} \middle| \sum_{k=1}^{n+1} (x_k)^2 = 1 \right\}.$$

Moreover, denote $\mathbb{S}^n \setminus \{(0,\ldots,-1)\}$ and $\mathbb{S}^n \setminus \{(0,\ldots,1)\}$, respectively, by H^n_+ and H^n_-. Define $y^n_+ : H^n_+ \longrightarrow \mathbb{R}^n$ and $y^n_- : H^n_- \longrightarrow \mathbb{R}^n$, respectively, by

$$y^n_+(x_1,\ldots,x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}},\ldots,\frac{x_n}{1+x_{n+1}}\right) \text{ for } (x_1,\ldots,x_{n+1}) \in H^n_+, \text{ and}$$
$$y^n_-(x_1,\ldots,x_{n+1}) = \left(\frac{x_1}{1-x_{n+1}},\ldots,\frac{x_n}{1-x_{n+1}}\right) \text{ for } (x_1,\ldots,x_{n+1}) \in H^n_-.$$

Then $y^n_\pm$ form homeomorphisms. Thus, with $\left\{y^n_\pm\right\}$ acting as an atlas, $\mathbb{S}^n$ forms an n-dimensional topological manifold.

Remark. y^n_+ and y^n_-, respectively, are called the stereographic projections from $(0,\ldots,-1)$ and $(0,\ldots,1)$.

Example 1.2.2. Let M and N, respectively, be topological manifolds with atlases $\{x_\alpha\}_{\alpha\in\mathrm{A}}$ and $\{y_\beta\}_{\beta\in\mathrm{B}}$. Then, with $\{(x_\alpha,y_\beta)\}_{\alpha\in\mathrm{A},\beta\in\mathrm{B}}$ acting as an atlas, $M\times N$ forms a topological manifold of dimension $\dim(M)+\dim(N)$. A renowned and significant example ensues.

Denote by $\mathbb{T}^n$ the n-torus, videlicet,

$$\mathbb{T}^n=\prod_{k=1}^{n}\mathbb{S}^1.$$

Pursuant to the foregoing example, $\mathbb{T}^n$ forms an n-dimensional topological manifold.

Theorem 1.2.1 (Invariance of Dimension, Theorem 1.2 in [13])**.** Let M,N be topological manifolds. If M,N are homeomorphic, then $\dim(M)=\dim(N)$.

Definition 1.2.3. Let M be a topological manifold and x,y charts on M. Then
$y\circ x^{-1}\big|_{x(\mathrm{Dom}(x)\cap\mathrm{Dom}(y))}$ is called the transition map from x to y if $\mathrm{Dom}(x)\cap\mathrm{Dom}(y)\neq\varnothing$.

Remark. Naturally, $y\circ x^{-1}\big|_{x(\mathrm{Dom}(x)\cap\mathrm{Dom}(y))}$ forms a homeomorphism.

Definition 1.2.4. Let M be a topological manifold.

1. Let x,y be charts on M. Then x and y are said to be smoothly compatible if either $\mathrm{Dom}(x)\cap\mathrm{Dom}(y)=\varnothing$ or the transition maps between them are both C^∞.
2. Let $\mathscr{A}$ be an atlas for M. Then $\mathscr{A}$ is said to be smooth if any pair of its elements are smoothly compatible.

Example 1.2.3.

1. Naturally, $\{\mathrm{Id}_{\mathbb{R}^n}\}$ forms a smooth atlas for $\mathbb{R}^n$.
2. Pursuant to Example 1.2.1, $\{y^n_\pm\}$ forms a smooth atlas for $\mathbb{S}^n$.
3. Pursuant to Example 1.2.2, if $\{x_\alpha\}_{\alpha\in\mathrm{A}}$ and $\{y_\beta\}_{\beta\in\mathrm{B}}$ are smooth, then $\{(x_\alpha,y_\beta)\}_{\alpha\in\mathrm{A},\beta\in\mathrm{B}}$ forms a smooth atlas for $M\times N$. For instance, pursuant to the foregoing listing,
$\{(y^1_{\alpha_1},\ldots,y^1_{\alpha_n})\}_{\alpha_1,\ldots,\alpha_n\in\{\pm\}}$ forms a smooth atlas for $\mathbb{T}^n$.

Definition 1.2.5. Let M be a topological manifold and $\mathscr{A}$ a smooth atlas. Then $\mathscr{A}$ is said to be maximal if there is no smooth atlas properly containing it. In this case, $\mathscr{A}$ is called a smooth structure on M and $(M, \mathscr{A})$ a smooth manifold.

Remark. This constitutes the stage on which differential geometry performs.

Proposition 1.2.1 (Proposition 1.17 in [13])**.** Let M be a topological manifold and $\mathscr{A}$ a smooth atlas. Then there exists a unique maximal smooth atlas containing $\mathscr{A}$.

Remark. This unique atlas is called the smooth structure determined by $\mathscr{A}$.

Example 1.2.4. Pursuant to Example 1.2.3, equipped with the smooth structures determined by $\{\mathrm{Id}_{\mathbb{R}^n}\}$, $\{y^n_{\pm}\}$, and $\left\{\left(y^1_{\alpha_1}, \ldots, y^1_{\alpha_n}\right)\right\}_{\alpha_1,\ldots,\alpha_n \in \{\pm\}}$, respectively, $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{T}^n$ are naturally smooth manifolds.

Definition 1.2.6. Let $(M, \mathscr{A})$ be a smooth manifold. Then elements of $\mathscr{A}$ are called a smooth chart on $(M, \mathscr{A})$.

Definition 1.2.7. Let M, N be smooth manifolds and f an arbitrary map from M to N. Then f is said to be smooth if, for every $p \in M$, there exists a smooth chart x_p on M containing p and a smooth chart y_p on N containing $f(p)$ such that $y \circ f \circ x^{-1}$ is C^∞. The collection of all smooth maps from M to N will be denoted by $C^\infty(M, N)$.

Remark.

1. By virtue of the smooth compatibility between smooth charts, the definition is independent of the choice of x_p and y_p.
2. Equipped with the pointwise addition and multiplication, $C^\infty(M, \mathbb{R})$ forms a commutative real algebra, whereas $C^\infty(M, \mathbb{C})$ forms a commutative complex algebra.

Proposition 1.2.2 (Proposition 2.4 in [13])**.** Every smooth map is necessarily continuous.

Remark. This vindicates that Definition 1.2.7 does not contravene the common sense in $\mathbb{R}^n$ that smoothness implies continuity.

Definition 1.2.8. Let f be a smooth map between smooth manifolds. Then f is called a diffeomorphism if f is bijective and f^{-1} is smooth.

After defining smooth maps, we would like to differentiate them as we have always done in $\mathbb{R}^n$.

Definition 1.2.9. Let M be a smooth manifold and p a point of M.

1. Let v be an element of $\mathrm{Hom}_{\mathbb{R}}(C^\infty(M,\mathbb{R}),\mathbb{R})$. Then v is called a tangent vector to M at p if $v(fg) = v(f)g(p) + f(p)v(g)$ for all $f,g \in C^\infty(M,\mathbb{R})$.
2. The subspace of all tangent vectors to M at p is called the tangent space to M at p and will be denoted by T_pM.
3. $(T_pM)^*$ is called the cotangent space to M at p and will be denoted by T_p^*M.

Proposition 1.2.3 (Proposition 3.10 in [13])**.** Let M be a smooth manifold. Then $\dim(T_pM) = \dim(M)$ and, thus, $\dim\left(T_p^*M\right) = \dim(M)$ for all $p \in M$.

Definition 1.2.10. Let M be a smooth manifold.

1. $\coprod\limits_{p\in M} T_pM$ is called the tangent bundle of M and will be denoted by TM.
2. $\coprod\limits_{p\in M} T_p^*M$ is called the cotangent bundle of M and will be denoted by T^*M.

For lucidity and convenience, $\mathbb{F}$ will be adverted to as either $\mathbb{R}$ or $\mathbb{C}$ in the sequel.

Definition 1.2.11. Let M,E be topological spaces and π a continuous projection from E onto M. Then (E,π,M) is called an $\mathbb{F}$-vector bundle of rank m over M if, for every $p \in M$,

(a) $\pi^{-1}(p)$ is an m-dimensional $\mathbb{F}$-vector space, and

(b) there exist an open neighborhood U_p of p and a homeomorphism ϕ_p from $\pi^{-1}(U_p)$ to $U_p \times \mathbb{F}^m$ such that $\phi_p\Big|_{\pi^{-1}(q)}$ is an $\mathbb{F}$-linear isomorphism to $\{q\} \times \mathbb{F}^m \equiv \mathbb{F}^m$ for all $q \in U_p$.

In this case, $\pi^{-1}(p)$ is called the fiber of (E,π,M) at p and (U_p,ϕ_p) a (local) trivialisation of (E,π,M) over U_p. Moreover, if X,E are smooth manifolds, π a smooth map, and ϕ_p a diffeomorphism, then (E,π,X) and (ϕ_p,U_p) are both said to be smooth.

Theorem 1.2.2 (Propositions 10.4, 11.9 in [13])**.** Let M be a smooth manifold. Then TM and T^*M are smooth real vector bundles of rank $\dim(M)$ over M.

Remark. This evinces why TM and T^*M, respectively, are entitled "tangent bundle" and "cotangent bundle".

Definition 1.2.12. Let (E,π,M) be a (smooth, respectively) vector bundle. Then (E,π,M) is said to be (smoothly, respectively) trivial if there exists a (smooth, respectively) (global) trivialisation of (E,π,M) over M.

Example 1.2.5 (Example 8.10 (a), (d), and Corollary 10.21 in [13])**.** $T\mathbb{R}^n$ and $T\mathbb{T}^n$ are smoothly trivial.

Definition 1.2.13. Let (E,π,M) be a (smooth, respectively) vector bundle and s a continuous (smooth, respectively) map from X to E. Then s is called a (smooth, respectively) section of (E,π,M) if $\pi \circ s = \mathrm{Id}_M$. The collection of all (smooth, respectively) sections of (E,π,M) will be denoted by $\Gamma(E)$ ($\Gamma^\infty(E)$, respectively).

Remark.

1. $\Gamma(E)$ ($\Gamma^\infty(E)$, respectively) forms a $C(M,\mathbb{F})$-module ($C^\infty(M,\mathbb{F})$-module, respectively). In particular, if (E,π,M) is (smoothly, respectively) trivial, then $\Gamma(E)$ ($\Gamma^\infty(E)$, respectively) is free.
2. If M is a smooth manifold, then $\Gamma^\infty(TM)$ and $\Gamma^\infty(T^*M)$, respectively, will be denoted by $\mathfrak{X}(M)$ and $\Omega(M)$.

Definition 1.2.14. Let M be a smooth manifold, (U,ϕ) a trivialisation of TM, and $\{E_k\}_{k=1}^{\dim(M)}$ a subset of $\Gamma(TM|_U)$. Then $\{E_k\}_{k=1}^{\dim(M)}$ is called a local frame for M if $\{E_k(p)\}_{k=1}^{\dim(M)}$ is a basis for T_pM for all $p \in U$.

Theorem 1.2.3 (Theorem 8.15 in [13])**.** Let M be a smooth manifold. Then $\mathfrak{X}(M)$ coincides with the real Lie algebra of all derivations on $C^\infty(M,\mathbb{R})$.

We now introduce the concept of Riemannian manifold, the protagonist of Riemannian geometry, and the concept of linear connection, the Excalibur that Riemannian manifold can be accoutred with.

Definition 1.2.15. Let M be a smooth manifold and g an $\mathbb{R}$-bilinear form on $\mathfrak{X}(M)$. Then g is called a Riemannian metric on M if, for every $p \in M$, $g(\cdot,\cdot)(p)$ is an inner product on T_pM, videlicet,

(a) $g(\cdot,X)$ is $C^\infty(M)$-linear for all $X \in \mathfrak{X}(M)$,

(b) $g(Y,X) = g(X,Y)$ for all $X,Y \in \mathfrak{X}(M)$,

(c) $g(X,X) \geq 0_M$ for all $X \in \mathfrak{X}(M)$, and

(d) $g(X,X)(p) = 0$ only if $X(p) = 0$.

In this case, (M,g) is called a Riemannian manifold.

Theorem 1.2.4 (Theorem 1.4.1 in [10])**.** Every smooth manifold can be invested with a Riemannian metric.

Definition 1.2.16. Let (M,g) be a Riemannian manifold and $\{E_k\}_{k=1}^{\dim(M)}$ a local frame. Then $\{E_k\}_{k=1}^{\dim(M)}$ is said to be orthonormal if $g(E_k,E_l) = \delta_{kl}$ for all $1 \leq k,l \leq \dim(M)$.

Definition 1.2.17. Let M be a smooth manifold and D an $\mathbb{R}$-bilinear map from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to $\mathfrak{X}(M)$. Then D is called a (linear) connection on M if

(a) $D(X,\cdot)$ is $C^\infty(M,\mathbb{R})$-linear for all $X \in \mathfrak{X}(M)$, and

(b) $D(fX,Y) = (Yf)X + fD(X,Y)$ for all $X,Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M,\mathbb{R})$.

In accordance with notational convention, $D(X,Y)$ will be denoted by D_YX.

Definition 1.2.18. Let M be a smooth manifold and D a connection on M. Define
$R^D : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathfrak{X}(M))$ by

$$R^D(X,Y) = D_XD_Y - D_YD_X - D_{[X,Y]} \text{ for } X,Y \in \mathfrak{X}(M).$$

Then R^D is called the curvature of D.

Proposition 1.2.4 (Theorem 3.1.2 in [10])**.** R^D is an element of $\mathrm{End}_{C^\infty(M,\mathbb{R})}(\mathfrak{X}(M)) \otimes_{C^\infty(M,\mathbb{R})} (\Omega(M) \wedge \Omega(M))$ (p.731 of [11]).

Definition 1.2.19. Let (M,g) be a Riemannian manifold and D a connection on M. Then D is said to be g-compatible if $Z(g(X,Y)) = g(D_Z X,Y) + g(X,D_Z Y)$ for all $X,Y,Z \in \mathfrak{X}(M)$.

Definition 1.2.20. Let M be a smooth manifold and D a connection on M. Then D is said to be torsion-free if $D_X Y - D_Y X - [X,Y] = 0$ for all $X,Y \in \mathfrak{X}(M)$.

Theorem 1.2.5 (Fundamental Theorem of Riemannian Geometry, Theorem 3.3.1 in [10])**.** Let (M,g) be a Riemannian manifold. Then there exists a unique g-compatible and torsion-free connection ∇ on (M,g), called the Levi-Civita connection. In particular, it is completely determined via

$$g(\nabla_Y X,Z) = \frac{1}{2}[Y(g(X,Z)) + X(g(Y,Z)) - Z(g(Y,X)) - g(Y,[X,Z]) - g(X,[Y,Z]) + g(Z,[Y,X])]$$
$$\text{for all } X,Y,Z \in \mathfrak{X}(M).$$

Proposition 1.2.5 (Lemma 3.3.1 in [10])**.** Let (M,g) be a Riemannian manifold.

1. $R^\nabla(X,Y)Z + R^\nabla(Y,Z)X + R^\nabla(Z,X)Y = 0$ for all $X,Y,Z \in \mathfrak{X}(M)$.
2. $R^\nabla(Y,X) = -R^\nabla(X,Y)$ for all $X,Y \in \mathfrak{X}(M)$.
3. $g\left(R^\nabla(X,Y)W,Z\right) = -g\left(R^\nabla(X,Y)Z,W\right)$ for all $X,Y,Z,W \in \mathfrak{X}(M)$.
4. $g\left(R^\nabla(Z,W)X,Y\right) = g\left(R^\nabla(X,Y)Z,W\right)$ for all $X,Y,Z,W \in \mathfrak{X}(M)$.

For the finale of our transient ramble, we shall state a renowned theorem that is apposite to our research. The most bewitching feature of the theorem resides in that it yokes the smooth structure and the topological structure on a compact manifold, relating the curvature of the Levi-Civita connection to a topological invariant.

Definition 1.2.21. Let M be a smooth manifold and $\mathscr{A}$ a smooth atlas.

1. Let x,y be elements of $\mathscr{A}$. Then x and y are said to be consistently oriented if either $\mathrm{Dom}(x) \cap \mathrm{Dom}(y) = \varnothing$ or the Jacobian determinants of the transition maps between them are positive.
2. $\mathscr{A}$ is said to be oriented if any pair of its elements are consistently oriented. In this case, $(M,\mathscr{A})$ is called an oriented manifold.
3. M is said to be orientable if there exists an oriented smooth atlas for M.

Example 1.2.6 (Example 15.22 in [13])**.** Though $\{y^n_\pm\}$ is not oriented, $\mathbb{S}^n$ is indeed orientable.

Proposition 1.2.6 (Proposition 15.17 and Corollary 10.21 in [13])**.** Every smooth manifold whose tangent bundle is smoothly trivial is necessarily orientable.

Example 1.2.7. Pursuant to Example 1.2.5, $\mathbb{R}^n$ and $\mathbb{T}^n$ are orientable.

Proposition 1.2.7 (Proposition 15.6 in [13])**.** Let M be a smooth manifold. If M is oriented, then T_pM is oriented for all $p \in M$.

Definition 1.2.22. Let M be an oriented manifold and $\{E_k\}_{k=1}^{\dim(M)}$ a local frame. Then $\{E_k\}_{k=1}^{\dim(M)}$ is said to be oriented if $\{E_k(p)\}_{k=1}^{\dim(M)}$ is oriented for all $p \in \mathrm{Dom}(E_1)$.

Proposition 1.2.8 (Proposition 15.29 in [13])**.** Let (M,g) be a Riemannian manifold. If M is oriented, then there exists a unique element ω_g in $\bigwedge\limits_{k=1}^{\dim(M)} \Omega(M)$, called the Riemannian volume form, such that $\omega_g(E_1,\ldots,E_{\dim(M)}) = 1_{\mathrm{Dom}(E_1)}$ for all oriented orthonormal local frames $\{E_k\}_{k=1}^{\dim(M)}$.

Definition 1.2.23. Let A be a skew-symmetric element of $\mathrm{M}_{2n}(\mathbb{R})$. Denote by S_{2n} the symmetric group on $2n$ letters (Definition 8.6 in [8]). Define $\mathrm{sgn} : S_{2n} \longrightarrow \{\pm 1\}$ by

$$\mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases} \quad \text{for } \sigma \in S_{2n} \text{ (Definition 9.18 in [8]).}$$

Then

$$\frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{k=1}^{n} A_{\sigma(2k-1),\sigma(2k)}$$

is called the Pfaffian of A and will be denoted by $\mathrm{Pf}(A)$.

Remark. We shall only utilise the first two cases.

1. If $n = 1$, then $\mathrm{Pf}(A) = A_{12}$.
2. If $n = 2$, then $\mathrm{Pf}(A) = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$.

Definition 1.2.24. Let (M,g) be a 2-dimensional Riemannian manifold, p a point of M, and $\{v_p, w_p\}$ a basis for T_pM. Then

$$\frac{g\left(R^{\nabla}(v_p,w_p)v_p,w_p\right)}{g(v_p,v_p)g(w_p,w_p)-g(v_p,w_p)^2}$$

is called the Gaussian curvature of M at p and will be denoted by $K(p)$.

Remark. By virtue of Propositions 1.2.3 and 1.2.4, the definition is independent of the choice of $\{v_p,w_p\}$.

Theorem 1.2.6 (Chern-Gauß-Bonnet Theorem, [5])**.** Let (M,g) be a $2n$-dimensional Riemannian manifold. Denote by $\chi(M)$ the Euler characteristic of M. If M is oriented and compact, then

$$\int_M \mathrm{Pf}\left(R^{\nabla}\right) = (2\pi)^n\chi(M).$$

Remark.

1. If $n=1$, then the theorem is equivalent to

$$\int_M K d\omega_g = -2\pi\chi(M).$$

2. $\chi(\mathbb{S}^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$.
3. $\chi(\mathbb{T}^n)=0$ for all $n\in\mathbb{N}$.

1.3 Noncommutative Geometry

After exploring in both the territories of algebra and geometry, we now establish a bridge that can link the two areas.

Definition 1.3.1. Let $\mathcal{A}$ be a commutative complex algebra. Then the collection of all nonzero complex algebra homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ is called the spectrum of $\mathcal{A}$ and will be denoted by $\widehat{\mathcal{A}}$.

Proposition 1.3.1 (1.1.6 in [15])**.** Let $\mathcal{A}$ be a commutative Banach algebra. Then, equipped with the weak* topology, $\widehat{\mathcal{A}}$ is locally compact Hausdorff. Moreover, if $\mathcal{A}$ is unital, then $\widehat{\mathcal{A}}$ is compact.

Definition 1.3.2. Let $\mathcal{A}$ be a commutative complex algebra and a an element of $\mathcal{A}$. Then the evaluation map for $\widehat{\mathcal{A}}$ at a is called the Gelfand transform of a and will be denoted by $\widehat{a}$.

Theorem 1.3.1 (Gelfand Duality, Theorem 1.1.7 in [15])**.** Let $\mathcal{A}$ be a commutative C*-algebra. Then the Gelfand transform forms a C*-algebra isomorphism from $\mathcal{A}$ to $C_0\left(\widehat{\mathcal{A}}\right)$.

Remark. If $\mathcal{A}$ is unital, then $\widehat{\mathcal{A}}$ is compact and, thus, the Gelfand transform is an isomorphism from $\mathcal{A}$ to $C\left(\widehat{\mathcal{A}}\right)$.

Pursuant to the Gelfand Duality, to every commutative C*-algebra corresponds uniquely a locally compact Hausdorff space up to homeomorphisms, and vice versa. Thus, it suggests the notion that a noncommutative C*-algebra be identified with a yet nonexistent "noncommutative space", which constitutes the very spirit of noncommutative geometry. Especially, a unital C*-algebra would correspondingly be identified with a "noncommutative compact space".

Corollary 1.3.1. Let $\mathcal{A}$ be a C*-algebra and a a normal element. If $\mathcal{A}$ is unital, then the Gelfand transform forms an isomorphism from the C*-subalgebra generated by $\{a, \mathbb{1}_{\mathcal{A}}\}$ to $C(\sigma_{\mathcal{A}}(a))$. In particular, a is mapped to $\mathrm{Id}_{\sigma_{\mathcal{A}}(a)}$ under the Gelfand transform.

Remark. If $f \in C(\sigma_{\mathcal{A}}(a))$, then $f(a)$ will be adverted to as the inverse image of f under the Gelfand transform in the sequel.

Corollary 1.3.2 (Lemma 1.3.1 in [15])**.** Every positive element of a C*-algebra is self-adjoint.

Theorem 1.3.2. Let X be a locally compact Hausdorff space. Then $C_0(\cdot)$ forms a bijective map from the collection of all open subsets of X to that of all closed ideals of $C_0(X)$.

Thus, in accordance with the preceding ratiocination, the foregoing theorem further suggests that we identify a closed ideal of a noncommutative C*-algebra with an open subset of the corresponding "noncommutative space".

Definition 1.3.3. Let $\mathcal{A}$ be a C*-algebra and ϕ an element of $\mathcal{A}^*$. Then ϕ is said to be positive if ϕ maps positive elements to positive numbers.

Pursuant to Attribute 9 in Example 1.1.1, to every integration of elements in $C_0(X)$ corresponds uniquely a positive linear functional on $C_0(X)$, and vice versa. Thus, the concept of positive linear functional could be deemed a noncommutative analogue of that of integration of continuous functions.

Theorem 1.3.3 (Swan's Theorem, Theorem 2 in [21])**.** Let X be a compact Hausdorff space. Then $\Gamma(\cdot)$ forms a bijective map from the collection of all complex vector bundles over X to that of all finitely generated projective modules over $C(X)$.

Thus, in accordance with the ratiocination that succeeds Theorem 1.3.1, if $\mathcal{A}$ is a noncommutative unital C*-algebra, then Swan's Theorem suggests that we deem a left (or right) finitely generated projective module over $\mathcal{A}$ a complex vector bundle over the corresponding "noncommutative compact space".

To conclude this chapter and instigate the next ones, we introduce the concept of universal C*-algebra, to which the two classes of C*-algebras that will be concentrated on afterwards belong. Intrigued readers could refer to [1] or [2] for more details.

Let S be a set and denote by $\mathcal{F}^*(S)$ the free *-algebra determined by S. Moreover, let $\mathscr{R}$ be a subset of $\mathcal{F}^*(S)$ and denote by $\mathcal{I}^*(\mathscr{R})$ the *-ideal of $\mathcal{F}^*(S)$ generated by $\mathscr{R}$. Then $\mathcal{F}^*(S)/\mathcal{I}^*(\mathscr{R})$ is called the *-algebra determined by (the generators) S and (the relations) $\mathscr{R}$, and will be denoted by $\mathcal{A}^*(S|\mathscr{R})$. Define $\|\cdot\| : \mathcal{A}^*(S|\mathscr{R}) \longrightarrow [0,\infty]$ by

$$\|a\| = \sup\left\{\|\rho(a)\|_{\mathrm{op}} \middle| \rho \text{ is a representation of } \mathcal{A}^*(S|\mathscr{R})\right\} \text{ for } a \in \mathcal{A}^*(S|\mathscr{R}).$$

If $\mathrm{Im}(\|\cdot\|) \nsubseteq [0,\infty)$, then the universal C*-algebra determined by S and $\mathscr{R}$ does not exist, so universal objects need not exist in the category of C*-algebras. Notwithstanding, if $\mathrm{Im}(\|\cdot\|) \subseteq [0,\infty)$, then $\|\cdot\|$ forms a semi-norm on $\mathrm{Vect}_{\mathbb{C}}(\mathcal{A}^*(S|\mathscr{R}))$ that satisfies

(a) $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a,b \in \mathcal{A}^*(S|\mathscr{R})$,

(b) $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}^*(S|\mathscr{R})$, and

(c) $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}^*(S|\mathscr{R})$.

Denote $\{a \in \mathcal{A}^*(S|\mathscr{R}) : \|a\| = 0\}$ by $\mathcal{N}$. Then, by virtue of (a) and (b), $\mathcal{N}$ forms a *-ideal of $\mathcal{A}^*(S|\mathscr{R})$. Thus, $\|\cdot\|$ forms a C*-norm on $\mathcal{A}^*(S|\mathscr{R})/\mathcal{N}$ and the completion of $(\mathcal{A}^*(S|\mathscr{R})/\mathcal{N}, \|\cdot\|)$ forms a C*-algebra.

Definition 1.3.4. The completion of $(\mathcal{A}^*(S|\mathscr{R})/\mathcal{N}, \|\cdot\|)$ is called the universal C*-algebra determined by (the generators) S and (the relations) $\mathscr{R}$, and will be denoted by $C^*(S|\mathscr{R})$.

1.3.1 Example: Noncommutative Tori

Let $\Theta \in \mathrm{M}_n(\mathbb{R})$ with $\Theta^t = -\Theta$. Consider $S = \{U_k\}_{k=1}^n \cup \{1\}$ and

$$\mathscr{R} = \{U_k^* U_k - 1, U_k U_k^* - 1\}_{k=1}^n \cup \left\{U_l U_k - e^{i2\pi\Theta_{kl}} U_k U_l\right\}_{k,l=1}^n \cup \{1 \cdot U_k - U_k\}_{k=1}^n.$$

Then the construction that precedes Definition 1.3.4 does survive and, thus, $C^*(S|\mathscr{R})$ exists (p.193 of [18]).

Definition 1.3.5. $C^*(S|\mathscr{R})$ is called the n-dimensional noncommutative torus with parameter Θ and will be denoted by $C(\mathbb{T}_\Theta)$.

Remark.

1. 1 acts as the multiplicative identity.
2. U_k is unitary for all $1 \leq k \leq n$.
3. If $\Theta = 0_n$, then $C(\mathbb{T}_\Theta)$ coincides with $C(\mathbb{T}^n)$ (p.193 of [18]), which evinces, in accordance with the ratiocination that succeeds Theorem 1.3.1, why these C*-algebras are entitled "noncommutative tori".

$C(\mathbb{T}_\Theta)$ will constitute the main cast for Chapter 2. Some pertinent characteristics that will be utilised ensue.

Define $\alpha : \mathbb{T}^n \times (S \cup S^*) \longrightarrow C(\mathbb{T}_\Theta)$ by

$$\alpha_{(z_1,\ldots,z_n)}(U_k) = z_k U_k \text{ for } (z_1,\ldots,z_n) \in \mathbb{T}^n \text{ and } 1 \leq k \leq n,$$

$$\alpha_{(z_1,\ldots,z_n)}(1) = 1 \text{ for } (z_1,\ldots,z_n) \in \mathbb{T}^n, \text{ and}$$

$$\alpha_{(z_1,\ldots,z_n)}\circ^* =^* \circ\alpha_{(z_1,\ldots,z_n)} \text{ for } (z_1,\ldots,z_n)\in\mathbb{T}^n.$$

By direct computation, α is compatible with $\mathscr{R}$, videlicet,

1. $\alpha_{(\cdot)}(U_k^*)\alpha_{(\cdot)}(U_k) = 1$ and $\alpha_{(\cdot)}(U_k)\alpha_{(\cdot)}(U_k^*) = 1$ for all $1\le k\le n$, and
2. $\alpha_{(\cdot)}(U_l)\alpha_{(\cdot)}(U_k) = e^{i2\pi\Theta_{kl}}\alpha_{(\cdot)}(U_k)\alpha_{(\cdot)}(U_l)$ for all $1\le k,l\le n$.

Thus, $\alpha_{(\cdot)}$ can be extended from $S\cup S^*$ to $C(\mathbb{T}_\Theta)$, forming a Lie group action of $\mathbb{T}^n$ on $C(\mathbb{T}_\Theta)$. Specifically, α is ergodic in the sense that $\mathbb{C}\{1\}$ forms the fixed subset of $C(\mathbb{T}_\Theta)$ under α (p.193 of [18]).

Definition 1.3.6. Let a be an element of $C(\mathbb{T}_\Theta)$. Then a is said to be smooth if $\alpha_{(\cdot)}(a)$ is C^∞ with respect to the smooth structure on $\mathbb{T}^n$ and the norm on $C(\mathbb{T}_\Theta)$. The *-subalgebra of all smooth elements of $C(\mathbb{T}_\Theta)$ will be denoted by $C^\infty(\mathbb{T}_\Theta)$.

Remark. $C^\infty(\mathbb{T}_\Theta)$ is dense in $C(\mathbb{T}_\Theta)$ (p.193 of [18]).

Denote by $\mathcal{S}(\mathbb{Z}^n)$ the space of all complex Schwartz functions on $\mathbb{Z}^n$, videlicet,

$$\mathcal{S}(\mathbb{Z}^n) = \left\{\phi:\mathbb{Z}^n\to\mathbb{C}\,\middle|\, \sup_{(p_1,\ldots,p_n)\in\mathbb{Z}^n}(1+\sum_{k=1}^n p_k^2)^l|\phi(p_1,\ldots,p_n)|<\infty \text{ for all } l\in\mathbb{N}\right\}.$$

It has been widely known that there exists a unique $\mathbb{C}$-linear isomorphism $\mathcal{F}$ from $C^\infty(\mathbb{T}_\Theta)$ to $\mathcal{S}(\mathbb{Z}^n)$ such that

$$a = \sum_{(p_1,\ldots,p_n)\in\mathbb{Z}^n}\mathcal{F}(a)(p_1,\ldots,p_n)U_1^{p_1}\cdots U_n^{p_n} \text{ for all } a\in C^\infty(\mathbb{T}_\Theta)$$

(p.192 of [18]). Especially, $C^\infty(\mathbb{T}_\Theta)$ and $C^\infty(\mathbb{T}^n)$ are isomorphic as Fréchet spaces (Definition 1.1 in [19]).

Define $\tau: C^\infty(\mathbb{T}_\Theta)\longrightarrow\mathbb{C}$ by

$$\tau(a) = \mathcal{F}(a)(0,\ldots,0) \text{ for } a\in C^\infty(\mathbb{T}_\Theta).$$

Then τ forms a positive linear functional on $C^\infty(\mathbb{T}_\Theta)$ bounded by 1 (p.194 of [18]). Thus, τ can be uniquely extended to $C(\mathbb{T}_\Theta)$ because $C^\infty(\mathbb{T}_\Theta)$ is dense in $C(\mathbb{T}_\Theta)$ (p.194 of [18]). Specifically, τ is invariant under α because $\mathbb{C}\{1\}$ is the fixed subset of $C(\mathbb{T}_\Theta)$ under α. Moreover, if $\Theta = 0_n$, then τ coincides with the integration on $\mathbb{T}^n$ against the Haar measure (p.194 of [18]) and, thus, in accordance with the ratiocination that succeeds Definition 1.3.3, we will deem τ a noncommutative analogue of the Haar integration on $\mathbb{T}^n$.

From its definition, τ is tracial, videlicet,

$$\tau(ab) = \tau(ba) \text{ for all } a,b \in C(\mathbb{T}_\Theta).$$

If $n = 2$ and Θ_{12} is irrational, then $C(\mathbb{T}_\Theta)$ can be faithfully represented on $L^2\left(\mathbb{S}^1\right)$:

Example 1.3.1. Define $\rho : \{U_1, U_2, 1\} \longrightarrow \mathrm{End}_{\mathbb{C}}\left(L^2\left(\mathbb{S}^1\right)\right)$ by

$$\begin{gathered}
\rho(U_1)(\phi) = \mathrm{Id}_{\mathbb{S}^1}\phi \text{ for } \phi \in L^2\left(\mathbb{S}^1\right), \\
\rho(U_2)(\phi(\cdot)) = \phi\left(e^{i2\pi\Theta_{12}}\,\cdot\right) \text{ for } \phi \in L^2\left(\mathbb{S}^1\right), \text{ and} \\
\rho(1) = \mathrm{Id}_{L^2\left(\mathbb{S}^1\right)}.
\end{gathered}$$

Then $\rho(U_1)$ and $\rho(U_2)$ are unitary operators on $L^2\left(\mathbb{S}^1\right)$. Moreover, ρ is compatible with $\mathscr{R}$, videlicet, $\rho(U_2)\rho(U_1) = e^{i2\pi\Theta_{12}}\rho(U_1)\rho(U_2)$. Thus, ρ can be extended from $\{U_1, U_2, 1\}$ to $C(\mathbb{T}_\Theta)$, forming a *-algebra homomorphism from $C(\mathbb{T}_\Theta)$ to $\mathcal{B}\left(L^2\left(\mathbb{S}^1\right)\right)$. Specifically, if Θ_{12} is irrational, then ρ is injective because $C(\mathbb{T}_\Theta)$ is simple (p.600 of [6]).

1.3.2 Example: Quantum Discs and 2-Spheres

Let q be an element of $[0,1)$. Consider $S = \left\{z_q, 1\right\}$ and

$$\mathscr{R} = \left\{z_q^* z_q - q z_q z_q^* - (1-q)\right\} \cup \left\{1 \cdot z_q - z_q, z_q \cdot 1 - z_q, 1^* - 1\right\}.$$

Then the construction that precedes Definition 1.3.4 does survive and, thus, $C^*(S|\mathscr{R})$ exists (Definition 11 in [4] and II.8.3.2 (v) in [2]).

Definition 1.3.7. $C^*(S|\mathscr{R})$ is called the quantum disc with parameter q and will be denoted by $C\left(\overline{\mathbb{D}}_q\right)$.

Remark. 1 acts as the multiplicative identity.

$C\left(\overline{\mathbb{D}}_q\right)$ will constitute the main cast for Chapter 3. Some pertinent characteristics that will be utilised ensue.

Proposition 1.3.2 (Proposition 14 in [4] and II.8.3.2 (v) in [2])**.** Define $\sigma : S \longrightarrow C\left(\mathbb{S}^1\right)$ by

$$\sigma\left(z_q\right) = \mathrm{Id}_{\mathbb{S}^1} \text{ and } \sigma(1) = 1_{\mathbb{S}^1}.$$

Then σ can be extended from S to $C\left(\overline{\mathbb{D}}_q\right)$, forming a *-algebra homomorphism from $C\left(\overline{\mathbb{D}}_q\right)$ to $C\left(\mathbb{S}^1\right)$. Specifically, σ is surjective.

Denote by $\mathcal{H}_{\text{sep}}$ the separable infinite-dimensional Hilbert space. Moreover, fix an (ordered) orthonormal basis for $\mathcal{H}_{\text{sep}}$ in the sequel, say, $\{e_k\}_{k\in\mathbb{N}}$. Then, for every $k,l \in \mathbb{N}$, denote by e_{kl} the element of $\mathcal{B}\left(\mathcal{H}_{\text{sep}}\right)$ with $e_{kl}(e_m) = \delta_{lm}e_k$.

$C\left(\overline{\mathbb{D}}_q\right)$ can be faithfully represented on $\mathcal{H}_{\text{sep}}$:

Proposition 1.3.3 (Proposition 14 in [4] and II.8.3.2 (v) in [2])**.** Define $\gamma : S \longrightarrow \mathcal{B}\left(\mathcal{H}_{\text{sep}}\right)$ by

$$\gamma\left(z_q\right) = \sum_{k\in\mathbb{N}} \sqrt{1-q^k}\, e_{k+1,k} \text{ and } \gamma(1) = \mathbb{1}_{\mathcal{B}\left(\mathcal{H}_{\text{sep}}\right)}.$$

Then γ can be extended from S to $C\left(\overline{\mathbb{D}}_q\right)$, forming a *-algebra homomorphism from $C\left(\overline{\mathbb{D}}_q\right)$ to $\mathcal{B}\left(\mathcal{H}_{\text{sep}}\right)$. Specifically, γ is injective (p.380 of [4]).

Remark. $\gamma(z_0)$ is called the unilateral shift and $\gamma\left(C\left(\overline{\mathbb{D}}_0\right)\right)$ is called the Toeplitz algebra. Specifically, $\sigma_{\mathcal{B}\left(\mathcal{H}_{\text{sep}}\right)}(\gamma(z_0)) = \overline{\mathbb{D}}$ (Example 2.3.2 in [14]) and $\mathcal{K}\left(\mathcal{H}_{\text{sep}}\right)$ forms a closed ideal of $\gamma\left(C\left(\overline{\mathbb{D}}_0\right)\right)$ (II.8.3.2 (v) in [2]). Moreover, $\gamma\left(C\left(\overline{\mathbb{D}}_0\right)\right)/\mathcal{K}\left(\mathcal{H}_{\text{sep}}\right)$ is isomorphic to $C\left(\mathbb{S}^1\right)$ (II.8.3.2 (v) in [2]).

Proposition 1.3.4 (Proposition 15 in [4])**.** $C\left(\overline{\mathbb{D}}_q\right)$ is C*-algebraically isomorphic to $C\left(\overline{\mathbb{D}}_0\right)$ for all $q \in (0,1)$.

Remark. Pursuant to the foregoing remark, this evinces why these C*-algebras are entitled "quantum discs".

Corollary 1.3.3. $0 \to \mathcal{K}\left(\mathcal{H}_{\text{sep}}\right) \to C\left(\overline{\mathbb{D}}_q\right) \xrightarrow{\sigma} C\left(\mathbb{S}^1\right) \to 0$ is exact.

Proof. Pursuant to Proposition 1.3.4 and the remark that succeeds Proposition 1.3.3. $\square$

Remark. In accordance with the ratiocination that succeeds Theorem 1.3.2, $\mathcal{K}\left(\mathcal{H}_{\text{sep}}\right)$ would be identified with the "interior" (noncommutative open disc) of the quantum disc. Similarly, $C\left(\mathbb{S}^1\right)$ would be identified with the "boundary" of the quantum disc.

Let $(\mathcal{A},^{*_\mathcal{A}},\|\cdot\|_\mathcal{A}),(\mathcal{B},^{*_\mathcal{B}},\|\cdot\|_\mathcal{B})$ be C*-algebras. Extend componentwise $^{*_\mathcal{A}}$ and $^{*_\mathcal{B}}$, respectively, from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{A}\oplus\mathcal{B}$. Then $\mathcal{A}\oplus\mathcal{B}$ forms a *-algebra. Define $\|\cdot\| : \mathcal{A}\oplus\mathcal{B} \longrightarrow [0,\infty)$ by

$$\|(a,b)\| = \max\{\|a\|_\mathcal{A},\|b\|_\mathcal{B}\} \text{ for } (a,b)\in\mathcal{A}\oplus\mathcal{B}.$$

Then $(\mathcal{A}\oplus\mathcal{B},\|\cdot\|)$ forms a C*-algebra.

Definition 1.3.8. Let $(\mathcal{A},^{*_\mathcal{A}},\|\cdot\|_\mathcal{A}),(\mathcal{B},^{*_\mathcal{B}},\|\cdot\|_\mathcal{B})$ be C*-algebras. Then $(\mathcal{A}\oplus\mathcal{B},\|\cdot\|)$ is called the direct sum of $(\mathcal{A},^{*_\mathcal{A}},\|\cdot\|_\mathcal{A})$ and $(\mathcal{B},^{*_\mathcal{B}},\|\cdot\|_\mathcal{B})$, and will be denoted by $\mathcal{A}\oplus\mathcal{B}$.

Though there have been several different versions of quantum 2-spheres, we shall only focus on the subsequent two ones in this dissertation.

Definition 1.3.9 (Propositions 16, 21 in [4] and p.734 of [9])**.** Let q,p be elements of $[0,1)$.

1. $\left\{(a,b)\in C\left(\overline{\mathbb{D}}_q\right)\oplus C\left(\overline{\mathbb{D}}_p\right)\middle|\sigma(a)=\sigma(b)\right\}$, forming a C*-subalgebra of $C\left(\overline{\mathbb{D}}_q\right)\oplus C\left(\overline{\mathbb{D}}_p\right)$, is called the Podleś sphere with parameters q,p and will be denoted by $C\left(\mathbb{S}^2_{qp}\right)$.
2. $\left\{(a,b)\in C\left(\overline{\mathbb{D}}_q\right)\oplus C\left(\overline{\mathbb{D}}_p\right)\middle|\sigma(a)(\overline{\cdot})=\sigma(b)(\cdot)\right\}$, forming a C*-subalgebra of $C\left(\overline{\mathbb{D}}_q\right)\oplus C\left(\overline{\mathbb{D}}_p\right)$, is called the quantum complex projective line with parameters q,p and will be denoted by $C\left(\mathbb{CP}^1_{qp}\right)$. Specifically, since $\mathbb{CP}^1$ is homeomorphic to $\mathbb{S}^2$, $C\left(\mathbb{CP}^1_{qp}\right)$ is also considered a quantum 2-sphere.

Remark. In accordance with the remark that succeeds Corollary 1.3.3, we could deem these definitions noncommutative analogues of the connected sum that glues two closed discs to form a 2-sphere via identifying their boundary circles. In particular, the construction of the Podleś spheres amalgamates the quantum discs in such a way that they are "oppositely oriented", whereas that of the quantum complex projective line amalgamates the quantum discs in such a way that they are "consistently oriented".

Chapter 2

Riemannian Geometry on Noncommutative Tori

2.1 Noncommutative Levi-Civita Connections

We now commence transcribing Riemannian geometry for noncommutative tori, following step by step the framework developed in Rosenberg's paper [19], which was first conceived by Connes in his paper [6]. Though this primitive framework is rather simplistic compared with the more sophisticated but more complicated mechanism later developed by Connes based on spectral triples, Rosenberg did demonstrate successfully in his paper that interesting results can still be distilled therefrom, especially those that have counterparts in the commutative territory. Intrigued readers could refer to [19] and [16] for more details.

For every $1 \leq k \leq n$, denote by ∂_k the infinitesimal generator of α corresponding to the k-th coordinate (Subsection 1.3.1), videlicet,

$$\partial_k U_l = i2\pi\delta_{kl}U_k,\ 1 \leq l \leq n.$$

Moreover, denote by $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$ the free left $C^\infty(\mathbb{T}_\Theta)$-module spanned by $\{\partial_k\}_{k=1}^n$.

If $\Theta = 0_n$, then $C^\infty(\mathbb{T}_\Theta)$ and $C^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}}$, respectively, coincide with $C^\infty(\mathbb{T}^n,\mathbb{C})$ and $C^\infty(\mathbb{T}^n,\mathbb{R})$, and, thus, $C^\infty(\mathbb{T}_\Theta) \cong_{\mathrm{Alg}_\mathbb{C}} C^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}} \otimes_\mathbb{R} \mathbb{C}$. Thence, pursuant to Example 1.2.5 and the remark that succeeds Definition 1.2.13, $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$ coincides with $\mathfrak{X}(\mathbb{T}^n) \otimes_\mathbb{R} \mathbb{C}$, i.e. $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}^n\right)$. Therefore, in accordance with the ratiocination that succeeds Theorem 1.3.3, $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$ would be deemed the "(complexified) tangent bundle" of $C^\infty(\mathbb{T}_\Theta)$.

Nonetheless, if $\Theta \neq 0_n$, then, owing to the noncommutativity of $C^\infty(\mathbb{T}_\Theta)$, commutators in $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$ might not be derivations on $C^\infty(\mathbb{T}_\Theta)$ (p.2 of [19]). Thus, with the intention of generalising the concept of connection on $\mathbb{T}^n$, in which $\mathfrak{X}(\mathbb{T}^n)$ acts as a $C^\infty(\mathbb{T}^n,\mathbb{R})$-module and a Lie algebra, respectively, in the first and the second variables, we need another noncommutative analogue for $\mathfrak{X}(\mathbb{T}^n)$.

Definition 2.1.1. Let $(\mathcal{A},^*)$ be a *-algebra and δ a derivation on $\mathcal{A}$. Then δ is called a *-derivation on $(\mathcal{A},^*)$ if $\delta\circ^* =^* \circ\delta$.

Remark. Equipped with the commutator bracket, the collection of all *-derivations on $(\mathcal{A},^*)$ forms a real Lie algebra.

Denote by $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$ the real Lie algebra of all *-derivations on $C^\infty(\mathbb{T}_\Theta)$. Naturally, $\mathrm{Span}_{\mathbb{R}}\left(\{\partial_k\}_{k=1}^n\right)$ forms a commutative proper Lie subalgebra of $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$. Though $\{\partial_k\}_{k=1}^n$ does not generate $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$, the disparity consists in the Lie subalgebra of all inner *-derivations on $C^\infty(\mathbb{T}_\Theta)$.

Theorem 2.1.1 (Remark 4.3 in [3] and pp.3-4 of [16])**.** If Θ is "generically transcendental", then $\mathfrak{D}^\infty(\mathbb{T}_\Theta) = \mathrm{Span}_{\mathbb{R}}\left(\{\partial_k\}_{k=1}^n\right) \oplus \mathrm{ad}(iC^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}} \cap \mathrm{Ker}(\tau))$.

Remark. Since almost all Θ are "generically transcendental" (Remark 4.3 in [3]), we shall assume in the sequel that Θ is "generically transcendental". For other Θ, we may consider, without loss of generality, replacing $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$ with $\mathrm{Span}_{\mathbb{R}}\left(\{\partial_k\}_{k=1}^n\right) \oplus \mathrm{ad}(iC^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}} \cap \mathrm{Ker}(\tau))$.

We are now prepared to reproduce Riemannian geometry on $C^\infty(\mathbb{T}_\Theta)$.

Definition 2.1.2. Let g be an $\mathbb{R}$-bilinear form on $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$. Then g is called a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$ if g is a $C^\infty(\mathbb{T}_\Theta)$-valued inner product on $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$, videlicet,

(a) $g(\cdot,X)$ is $C^\infty(\mathbb{T}_\Theta)$-linear for all $X \in \Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$,

(b) $g(Y,X) = g(X,Y)^*$ for all $X,Y \in \Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$,

(c) $\left[g(\partial_k,\partial_l)\right]$ is an invertible and positive element of $\mathrm{M}_n(C^\infty(\mathbb{T}_\Theta))$, and

(d) $g(\partial_k,\partial_l)$ is self-adjoint for all $1 \le k,l \le n$.

Remark. In the classical case, i.e. $\Theta = 0_n$, $g(\partial_k,\partial_l)$ assumes values in $\mathbb{R}$ and, thus, is self-adjoint, which evinces why condition (d) is imposed.

Definition 2.1.3. Let D be an $\mathbb{R}$-bilinear map from $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right) \times \mathfrak{D}^\infty(\mathbb{T}_\Theta)$ to $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$. Then D is called a (linear) connection on $C^\infty(\mathbb{T}_\Theta)$ if

(a) $D(aX,Y) = (Ya)X + aD(X,Y)$ for all $X \in \Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$, $Y \in \mathfrak{D}^\infty(\mathbb{T}_\Theta)$, and $a \in C^\infty(\mathbb{T}_\Theta)$,

(b) $D(X, \mathrm{ad}(a)) = aX$ for all $X \in \Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$ and $a \in iC^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}} \cap \mathrm{Ker}(\tau)$ (pp.3-4 of [16]), and

(c) $g(D(\partial_k, \partial_l), \partial_m)$ is self-adjoint for all $1 \leq k,l,m \leq n$.

In accordance with notational convention, $D(X,Y)$ will be denoted by $D_Y X$.

Remark.

1. By virtue of the bijectivity of $\mathrm{ad}\big|_{iC^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}} \cap \mathrm{Ker}(\tau)}$ (pp.3-4 of [16]), condition (b) is well-defined.
2. In the classical case, i.e. $\Theta = 0_n$, $g(D(\partial_k, \partial_l), \partial_m)$ assumes values in $\mathbb{R}$ and, thus, is self-adjoint, which evinces why condition (c) is imposed.

Definition 2.1.4. Let D be a connection on $C^\infty(\mathbb{T}_\Theta)$. Define
$R^D : \mathfrak{D}^\infty(\mathbb{T}_\Theta) \times \mathfrak{D}^\infty(\mathbb{T}_\Theta) \longrightarrow \mathrm{End}_{\mathbb{C}}\left(\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)\right)$ by

$$R^D(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]} \text{ for } X,Y \in \mathfrak{D}^\infty(\mathbb{T}_\Theta).$$

Then R^D is called the curvature of D.

Remark. Naturally, R^D is $\mathbb{R}$-bilinear.

Proposition 2.1.1 (Proposition 3.3 in [19])**.** Let D be a connection on $C^\infty(\mathbb{T}_\Theta)$. Then $R^D(X,Y)$ belongs to $\mathrm{End}_{C^\infty(\mathbb{T}_\Theta)}\left(\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)\right)$ for all $X,Y \in \mathfrak{D}^\infty(\mathbb{T}_\Theta)$.

Remark. This suggests the notion that $R^D(X,Y)$ be deemed a "noncommutative $(1,1)$-tensor".

Proposition 2.1.2 (Remark 3.2 in [19] and pp.3-4 of [16])**.** Let D be a connection on $C^\infty(\mathbb{T}_\Theta)$. Then $R^D(\mathrm{ad}(a), \cdot) \equiv 0$ for all $a \in iC^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}} \cap \mathrm{Ker}(\tau)$.

Remark. If condition (b) in Definition 2.1.3 is superseded by

$$D_{\mathrm{ad}(a)} X = aX + \mu(X) \text{ for some } \mu \in \mathrm{Hom}_{\mathbb{R}}\left(iC^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}} \cap \mathrm{Ker}(\tau), \mathrm{End}_{C^\infty(\mathbb{T}_\Theta)}\left(\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)\right)\right),$$

then $R^D(\mathrm{ad}(a), \cdot)$ might not be zero identically (Proposition 2 in [16]).

Definition 2.1.5. Let g and D, respectively, be a Riemannian metric and a connection on $C^\infty(\mathbb{T}_\Theta)$. Then D is said to be g-compatible if $Z(g(X,Y)) = g(D_Z X, Y) + g(X, D_Z Y)$ for all $X, Y \in \Gamma^\infty(T^{\mathbb{C}}\mathbb{T}_\Theta)$ and $Z \in \mathfrak{D}^\infty(\mathbb{T}_\Theta)$.

Definition 2.1.6. Let D be a connection on $C^\infty(\mathbb{T}_\Theta)$. Then D is said to be torsion-free if $D_{\partial_l}\partial_k - D_{\partial_k}\partial_l = 0$ for all $1 \leq k, l \leq n$.

Theorem 2.1.2 (Theorem 2.1 in [19])**.** Let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$. Then there exists a unique g-compatible and torsion-free connection ∇ on $(C^\infty(\mathbb{T}_\Theta), g)$, called the Levi-Civita connection. In particular, it is completely determined via

$$g\left(\nabla_{\partial_j}\partial_i, \partial_k\right) = \frac{1}{2}\left(\partial_j g(\partial_i, \partial_k) + \partial_i g\left(\partial_j, \partial_k\right) - \partial_k g\left(\partial_j, \partial_i\right)\right) \text{ for all } 1 \leq i, j, k \leq n.$$

Remark. If condition (b) in Definition 2.1.3 is relaxed, then

$$\nabla + \mathrm{Hom}_{\mathbb{R}}\left(iC^\infty(\mathbb{T}_\Theta)_{\mathrm{sa}} \cap \mathrm{Ker}(\tau), \mathrm{Ker}\left({}^* + \mathrm{Ad}(g)^{-1}\right)\right)$$

forms a family of g-compatible and torsion-free connections on $C^\infty(\mathbb{T}_\Theta)$ (Proposition 1 in [16]).

Proposition 2.1.3 (Proposition 3.4 in [19])**.** Let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$.

1. $R^\nabla(X,Y)Z + R^\nabla(Y,Z)X + R^\nabla(Z,X)Y = 0$ for all $X, Y, Z \in \mathrm{Span}_{\mathbb{R}}\left(\{\partial_k\}_{k=1}^n\right)$.
2. $R^\nabla(Y,X) = -R^\nabla(X,Y)$ for all $X, Y \in \mathfrak{D}^\infty(\mathbb{T}_\Theta)$.

Remark. These are parallels to results in Proposition 1.2.5. Notwithstanding, listing 3 in Proposition 1.2.5 does not hold in noncommutative territory, i.e. $g\left(R^\nabla(X,Y)Z, W\right) \neq -g\left(R^\nabla(X,Y)W, Z\right)$ in general (p.7 of [19]) and, thus, listing 4 does not either. In particular, $g\left(R^\nabla(X,Y)Z, Z\right)$ might not equal zero.

For lucidity and convenience, notational conventions from general relativity will be adopted, videlicet, in the sequel,

(a) $g(\partial_k, \partial_l)$ will be denoted by g_{kl},

(b) $g\left(R^\nabla\left(\partial_i, \partial_j\right)\partial_k, \partial_l\right)$ will be denoted by R_{lkij},

(c) $\partial_{k_m} \cdots \partial_{k_1} a$ will be denoted by $a_{,k_1 \cdots k_m}$, and

(d) $\nabla_{\partial_k} X$ will be denoted by $X_{;k}$.

Moreover, the Einstein summation convention will also be adopted, scilicet, if an index in a term occurs twice with one in superscript and the other in subscript, then the term shall be summed over all possible values of the index; for instance, $\sum\limits_{k=1}^{n} a^k \partial_k$ will be denoted by $a^k \partial_k$.

Two adaptations of the Chern-Gauß-Bonnet Theorem have been established for $n = 2$.

Theorem 2.1.3 (Proposition 4.1 in [19])**.** For $n = 2$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[g_{kl}\right] = \mathrm{Diag}(E, E)$$

for some $E \in C^\infty(\mathbb{T}_\Theta)_\geq$. Then

$$\tau\left(R_{2112} E^{-1}\right) = 0.$$

Theorem 2.1.4 (Theorem in [16])**.** For $n = 2$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[g_{kl}\right] = \mathrm{Diag}(E, G)$$

for some $E, G \in C^\infty(\mathbb{T}_\Theta)_\geq$.

1. If $E_{,k} E = E E_{,k}$ and $G_{,k} G = G G_{,k}$ for all $1 \leq k \leq 2$, then

$$\tau\left(E^{-\frac{1}{2}} R_{2112} G^{-\frac{1}{2}}\right) = 0.$$

2. If $EG = 1$, then

$$\tau\left(E^{-\frac{1}{2}} R_{2112} G^{-\frac{1}{2}}\right) = 0.$$

Remark.

1. $(E, G) = (U_1 + U_1^* + 3, U_2 + U_2^* + 3)$ constitutes a nontrivial instance in the sense that G does not commute with E (p.10 of [16]).

2. Since τ is deemed a noncommutative analogue of the Haar integration (Subsection 1.3.1), $\tau(\,\cdot\,E)$ and $\tau\left(E^{\frac{1}{2}}(\cdot)G^{\frac{1}{2}}\right)$ should be identified with the corresponding integrations against the Riemannian volume form. Thus, in accordance with the remark that succeeds Theorem 1.2.6, we could view Theorems 2.1.3 and 2.1.4 as noncommutative variants of Theorem 1.2.6 on $C^\infty(\mathbb{T}_\Theta)$.

2.2 2-dimensional Cases

Some extensions of Theorems 2.1.3 and 2.1.4 will be tendered in this section. We first augment the formula in Theorem 2.1.2.

Proposition 2.2.1. Let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$ and $\{X_k\}_{k=1}^n$ a subset of $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$[X_k] = [S_k{}^l \partial_l]$$

for some $S \in \mathrm{GL}_n(\mathbb{R})$. Then

$$g\left(\nabla_{X_j} X_i, X_k\right) = \frac{1}{2}\left(X_j g(X_i, X_k) + X_i g\left(X_j, X_k\right) - X_k g\left(X_j, X_i\right)\right)$$

for all $1 \le i, j, k \le n$.

Proof.

$$\begin{aligned}
& g\left(\nabla_{X_j} X_i, X_k\right) \\
&= g\left(\nabla_{S_j{}^\beta \partial_\beta} S_i{}^\alpha \partial_\alpha, S_k{}^\gamma \partial_\gamma\right) \\
&= g\left(S_i{}^\alpha S_j{}^\beta \partial_{\alpha;\beta}, S_k{}^\gamma \partial_\gamma\right) && (\nabla \text{ is } \mathbb{R}\text{-bilinear}) \\
&= S_i{}^\alpha S_j{}^\beta S_k{}^\gamma g\left(\partial_{\alpha;\beta}, \partial_\gamma\right) && (g \text{ is } \mathbb{R}\text{-bilinear}) \\
&= S_i{}^\alpha S_j{}^\beta S_k{}^\gamma \cdot \frac{1}{2}\left(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\beta\alpha,\gamma}\right) && (\text{by Theorem 2.1.2}) \\
&= \frac{1}{2}\left(S_j{}^\beta \left(S_i{}^\alpha g_{\alpha\gamma} S_k{}^\gamma\right)_{,\beta} + S_i{}^\alpha \left(S_j{}^\beta g_{\beta\gamma} S_k{}^\gamma\right)_{,\alpha} - S_k{}^\gamma \left(S_j{}^\beta g_{\beta\alpha} S_i{}^\alpha\right)_{,\gamma}\right) && ([\partial_k] \text{ is } \mathbb{R}\text{-linear}) \\
&= \frac{1}{2}\left(X_j g(X_i, X_k) + X_i g\left(X_j, X_k\right) - X_k g\left(X_j, X_i\right)\right).
\end{aligned}$$

□

The subsequent results are proved utilising the methods demonstrated in [19] and [16]. We explicitly exhibit those proofs here for the sake of coherence and clarity.

Proposition 2.2.2. For $n=2$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$ and $\{X_k\}_{k=1}^2$ a subset of $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[X_k\right] = \left[S_k{}^l \partial_l\right] \text{ and } \left[g(X_k,X_l)\right] = \mathrm{Diag}(E,G)$$

respectively, for some $S \in \mathrm{GL}_2(\mathbb{R})$ and some $E,G \in C^\infty(\mathbb{T}_\Theta)_{\geq}$. Then

$$\begin{gathered} g\left(R^\nabla(X_1,X_2)X_1,X_2\right) = \\ \frac{1}{2}\left(-\frac{1}{2}(X_2E)E^{-1}(X_2E) + X_2^2E - \frac{1}{2}(X_2E)G^{-1}(X_2G)\right) + \\ \frac{1}{2}\left(-\frac{1}{2}(X_1E)E^{-1}(X_1G) + X_1^2G - \frac{1}{2}(X_1G)G^{-1}(X_1G)\right), \text{ and} \end{gathered}$$

$$\begin{gathered} g\left(R^\nabla(X_1,X_2)X_1,X_1\right) = \\ \frac{1}{4}\left((X_1E)E^{-1}(X_2E) - (X_2E)E^{-1}(X_1E)\right) + \frac{1}{4}\left((X_1G)G^{-1}(X_2E) - (X_2E)G^{-1}(X_1G)\right) \text{ and} \\ g\left(R^\nabla(X_1,X_2)X_2,X_2\right) = \\ \frac{1}{4}\left((X_1G)G^{-1}(X_2G) - (X_2G)G^{-1}(X_1G)\right) + \frac{1}{4}\left((X_1G)E^{-1}(X_2E) - (X_2E)E^{-1}(X_1G)\right). \end{gathered}$$

Proof. Pursuant to the metric hypothesis and Proposition 2.2.1,

$$\begin{aligned} g(\nabla_{X_1}X_1,X_1) &= \frac{1}{2}X_1E, & g(\nabla_{X_1}X_1,X_2) &= -\frac{1}{2}X_2E, \\ g(\nabla_{X_2}X_1,X_1) &= \frac{1}{2}X_2E, & g(\nabla_{X_2}X_1,X_2) &= \frac{1}{2}X_1G, \\ g(\nabla_{X_1}X_2,X_2) &= \frac{1}{2}X_1G, & g(\nabla_{X_1}X_2,X_1) &= \frac{1}{2}X_2E, \\ g(\nabla_{X_2}X_2,X_2) &= \frac{1}{2}X_2G, & g(\nabla_{X_2}X_2,X_1) &= -\frac{1}{2}X_1G. \end{aligned}$$

Thus, since $\{X_k\}_{k=1}^2$ is an orthogonal basis for $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$,

$$\begin{aligned} \nabla_{X_1}X_1 &= \frac{1}{2}(X_1E)E^{-1}X_1 - \frac{1}{2}(X_2E)G^{-1}X_2, & \nabla_{X_2}X_1 &= \frac{1}{2}(X_2E)E^{-1}X_1 + \frac{1}{2}(X_1G)G^{-1}X_2, \\ \nabla_{X_1}X_2 &= \frac{1}{2}(X_1G)G^{-1}X_2 + \frac{1}{2}(X_2E)E^{-1}X_1, & \nabla_{X_2}X_2 &= \frac{1}{2}(X_2G)G^{-1}X_2 - \frac{1}{2}(X_1G)E^{-1}X_1. \end{aligned}$$

Thence,

$$\begin{aligned} &g\left(R^\nabla(X_1,X_2)X_1,X_2\right) \\ &= g(\nabla_{X_1}(\nabla_{X_2}X_1) - \nabla_{X_2}(\nabla_{X_1}X_1),X_2) \end{aligned}$$

$$
\begin{aligned}
&= g(\nabla_{X_1}(\nabla_{X_2}X_1),X_2) - g(\nabla_{X_2}(\nabla_{X_1}X_1),X_2) \\
&= g\left(\nabla_{X_1}\left(\frac{1}{2}(X_2E)E^{-1}X_1 + \frac{1}{2}(X_1G)G^{-1}X_2\right),X_2\right) - \\
&\quad g\left(\nabla_{X_2}\left(\frac{1}{2}(X_1E)E^{-1}X_1 - \frac{1}{2}(X_2E)G^{-1}X_2\right),X_2\right) \\
&= \frac{1}{2}g\left(\nabla_{X_1}\left((X_2E)E^{-1}X_1\right),X_2\right) + \frac{1}{2}g\left(\nabla_{X_1}\left((X_1G)G^{-1}X_2\right),X_2\right) - \\
&\quad \frac{1}{2}g\left(\nabla_{X_2}\left((X_1E)E^{-1}X_1\right),X_2\right) + \frac{1}{2}g\left(\nabla_{X_2}\left((X_2E)G^{-1}X_2\right),X_2\right) \\
&= \frac{1}{2}\left(0 - \frac{1}{2}(X_2E)E^{-1}(X_2E)\right) + \frac{1}{2}\left(\left(X_1^2G - (X_1G)G^{-1}(X_1G)\right) + \frac{1}{2}(X_1G)G^{-1}(X_1G)\right) - \\
&\quad \frac{1}{2}\left(0 + \frac{1}{2}(X_1E)E^{-1}(X_1G)\right) + \frac{1}{2}\left(\left(X_2^2E - (X_2E)G^{-1}(X_2G)\right) + \frac{1}{2}(X_2E)G^{-1}(X_2G)\right) \\
&= \frac{1}{2}\left(-\frac{1}{2}(X_2E)E^{-1}(X_2E) + X_2^2E - \frac{1}{2}(X_2E)G^{-1}(X_2G)\right) + \\
&\quad \frac{1}{2}\left(-\frac{1}{2}(X_1E)E^{-1}(X_1G) + X_1^2G - \frac{1}{2}(X_1G)G^{-1}(X_1G)\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&g\left(R^{\nabla}(X_1,X_2)X_1,X_1\right) \\
&= g(\nabla_{X_1}(\nabla_{X_2}X_1) - \nabla_{X_2}(\nabla_{X_1}X_1),X_1) \\
&= g(\nabla_{X_1}(\nabla_{X_2}X_1),X_1) - g(\nabla_{X_2}(\nabla_{X_1}X_1),X_1) \\
&= g\left(\nabla_{X_1}\left(\frac{1}{2}(X_2E)E^{-1}X_1 + \frac{1}{2}(X_1G)G^{-1}X_2\right),X_1\right) - \\
&\quad g\left(\nabla_{X_2}\left(\frac{1}{2}(X_1E)E^{-1}X_1 - \frac{1}{2}(X_2E)G^{-1}X_2\right),X_1\right) \\
&= \frac{1}{2}g\left(\nabla_{X_1}\left((X_2E)E^{-1}X_1\right),X_1\right) + \frac{1}{2}g\left(\nabla_{X_1}\left((X_1G)G^{-1}X_2\right),X_1\right) - \\
&\quad \frac{1}{2}g\left(\nabla_{X_2}\left((X_1E)E^{-1}X_1\right),X_1\right) + \frac{1}{2}g\left(\nabla_{X_2}\left((X_2E)G^{-1}X_2\right),X_1\right) \\
&= \frac{1}{2}\left(\left(X_1X_2E - (X_2E)E^{-1}(X_1E)\right) + \frac{1}{2}(X_2E)E^{-1}(X_1E)\right) + \frac{1}{2}\left(0 + \frac{1}{2}(X_1G)G^{-1}(X_2E)\right) - \\
&\quad \frac{1}{2}\left(\left(X_2X_1E - (X_1E)E^{-1}(X_2E)\right) + \frac{1}{2}(X_1E)E^{-1}(X_2E)\right) + \frac{1}{2}\left(0 - \frac{1}{2}(X_2E)G^{-1}(X_1G)\right) \\
&= \frac{1}{4}\left((X_1E)E^{-1}(X_2E) - (X_2E)E^{-1}(X_1E)\right) + \frac{1}{4}\left((X_1G)G^{-1}(X_2E) - (X_2E)G^{-1}(X_1G)\right).
\end{aligned}
$$

Similarly,

$$
g\left(R^{\nabla}(X_1,X_2)X_2,X_2\right)
$$

$$= \frac{1}{4}\left((X_1G)G^{-1}(X_2G) - (X_2G)G^{-1}(X_1G)\right) + \frac{1}{4}\left((X_1G)E^{-1}(X_2E) - (X_2E)E^{-1}(X_1G)\right).$$

□

Corollary 2.2.1. For $n = 2$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$ and $\{X_k\}_{k=1}^2$ a subset of $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[X_k\right] = \left[S_k{}^l \partial_l\right] \text{ and } \left[g(X_k, X_l)\right] = \mathrm{Diag}(E, E)$$

respectively, for some $S \in \mathrm{GL}_2(\mathbb{R})$ and some $E \in C^\infty(\mathbb{T}_\Theta)_\geq$. Then

$$\tau\left(g\left(R^\nabla(X_1,X_2)X_1,X_2\right)E^{-1}\right) = 0.$$

Proof. Pursuant to the metric hypothesis and Proposition 2.2.2,

$$\tau\left(g\left(R^\nabla(X_1,X_2)X_1,X_2\right)E^{-1}\right)$$
$$= \frac{1}{2}\tau\left(\left(-\frac{1}{2}(X_2E)E^{-1}(X_2E) + X_2^2E - \frac{1}{2}(X_2E)E^{-1}(X_2E)\right)E^{-1}\right) +$$
$$\frac{1}{2}\tau\left(\left(-\frac{1}{2}(X_1E)E^{-1}(X_1E) + X_1^2E - \frac{1}{2}(X_1E)E^{-1}(X_1E)\right)E^{-1}\right)$$
$$= \frac{1}{2}\tau\left(X_2\left((X_2E)E^{-1}\right)\right) + \frac{1}{2}\tau\left(X_1\left((X_1E)E^{-1}\right)\right)$$
$$= \frac{1}{2}\tau\left(S_2{}^k\left((X_2E)E^{-1}\right)_{,k}\right) + \frac{1}{2}\tau\left(S_1{}^k\left((X_1E)E^{-1}\right)_{,k}\right)$$
$$= \frac{1}{2}S_2{}^k\tau\left(\left((X_2E)E^{-1}\right)_{,k}\right) + \frac{1}{2}S_1{}^k\tau\left(\left((X_1E)E^{-1}\right)_{,k}\right).$$

Thus, since τ is invariant under the group action generated by $\{\partial_k\}_{k=1}^2$,

$$\tau\left(g\left(R^\nabla(X_1,X_2)X_1,X_2\right)E^{-1}\right) = 0 + 0 = 0.$$

□

Corollary 2.2.2. For $n = 2$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$ and $\{X_k\}_{k=1}^2$ a subset of $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[X_k\right] = \left[S_k{}^l \partial_l\right] \text{ and } \left[g(X_k, X_l)\right] = \mathrm{Diag}(E, G)$$

respectively, for some $S \in \mathrm{GL}_2(\mathbb{R})$ and some $E, G \in C^\infty(\mathbb{T}_\Theta)_\geq$. If $(X_kE)E = E(X_kE)$ and $(X_kG)G = G(X_kG)$ for all $1 \leq k \leq 2$, then

$$\tau\left(E^{-\frac{1}{2}}g\left(R^\nabla(X_1,X_2)X_1,X_2\right)G^{-\frac{1}{2}}\right) = 0.$$

Proof. Pursuant to the commutativity hypotheses and Proposition 2.2.2,

$$
\begin{aligned}
&\tau\left(E^{-\frac{1}{2}}g\left(R^{\nabla}(X_1,X_2)X_1,X_2\right)G^{-\frac{1}{2}}\right)\\
&=\frac{1}{2}\tau\left(E^{-\frac{1}{2}}\left(-\frac{1}{2}(X_2E)E^{-1}(X_2E)+X_2^2E-\frac{1}{2}(X_2E)G^{-1}(X_2G)\right)G^{-\frac{1}{2}}\right)+\\
&\quad\frac{1}{2}\tau\left(E^{-\frac{1}{2}}\left(-\frac{1}{2}(X_1E)E^{-1}(X_1G)+X_1^2G-\frac{1}{2}(X_1G)G^{-1}(X_1G)\right)G^{-\frac{1}{2}}\right)\\
&=\frac{1}{2}\tau\left(X_2\left(E^{-\frac{1}{2}}(X_2E)G^{-\frac{1}{2}}\right)\right)+\frac{1}{2}\tau\left(X_1\left(E^{-\frac{1}{2}}(X_1G)G^{-\frac{1}{2}}\right)\right)\\
&=\frac{1}{2}{S_2}^k\tau\left(\left(E^{-\frac{1}{2}}(X_2E)G^{-\frac{1}{2}}\right)_{,k}\right)+\frac{1}{2}{S_1}^k\tau\left(\left(E^{-\frac{1}{2}}(X_1G)G^{-\frac{1}{2}}\right)_{,k}\right).
\end{aligned}
$$

Thus, since τ is invariant under the group action generated by $\{\partial_k\}_{k=1}^2$,

$$\tau\left(E^{-\frac{1}{2}}g\left(R^{\nabla}(X_1,X_2)X_1,X_2\right)G^{-\frac{1}{2}}\right)=0+0=0.$$

□

Remark. In accordance with the remark that succeeds Theorem 2.1.4, Corollaries 2.2.1 and 2.2.2 could also be viewed as noncommutative variants of Theorem 1.2.6 on $C^\infty(\mathbb{T}_\Theta)$.

Though R_{kk12} does not equal zero in general (the remark that succeeds Proposition 2.1.3), their "integrals" do in Theorems 2.1.3 and 2.1.4 (with some additional commutativity assumptions for Theorem 2.1.4).

Corollary 2.2.3. For $n=2$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$ and $\{X_k\}_{k=1}^2$ a subset of $\mathfrak{D}^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[X_k\right]=\left[{S_k}^l\partial_l\right] \text{ and } \left[g(X_k,X_l)\right]=\mathrm{Diag}(E,E)$$

respectively, for some $S\in\mathrm{GL}_2(\mathbb{R})$ and some $E\in C^\infty(\mathbb{T}_\Theta)_{\geq}$. Then

$$\tau\left(g\left(R^{\nabla}(X_1,X_2)X_k,X_k\right)E^{-1}\right)=0$$

for all $1\leq k\leq 2$.

Proof. Pursuant to the metric hypothesis and Proposition 2.2.2,

$$
\begin{aligned}
&\tau\left(g\left(R^{\nabla}(X_1,X_2)X_1,X_1\right)E^{-1}\right)\\
&=\frac{1}{4}\tau\left(\left((X_1E)E^{-1}(X_2E)-(X_2E)E^{-1}(X_1E)\right)E^{-1}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4}\tau\left(\left((X_1E)E^{-1}(X_2E)-(X_2E)E^{-1}(X_1E)\right)E^{-1}\right)\\
&=\frac{1}{4}\tau\left((X_1E)E^{-1}(X_2E)E^{-1}\right)-\frac{1}{4}\tau\left((X_1E)E^{-1}(X_2E)E^{-1}\right)+\\
&\quad\frac{1}{4}\tau\left((X_1E)E^{-1}(X_2E)E^{-1}\right)-\frac{1}{4}\tau\left((X_1E)E^{-1}(X_2E)E^{-1}\right) && (\tau \text{ is tracial})\\
&=0.
\end{aligned}
$$

Similarly, $\tau\left(g\left(R^{\nabla}(X_1,X_2)X_2,X_2\right)E^{-1}\right)=0$. $\square$

Corollary 2.2.4. For $n=2$, let g be a Riemannian metric on $C^{\infty}(\mathbb{T}_{\Theta})$ and $\{X_k\}_{k=1}^{2}$ a subset of $\mathfrak{D}^{\infty}(\mathbb{T}_{\Theta})$. Suppose further that

$$
\left[X_k\right]=\left[S_k{}^{l}\partial_l\right] \text{ and } \left[g(X_k,X_l)\right]=\mathrm{Diag}(E,G)
$$

respectively, for some $S\in\mathrm{GL}_2(\mathbb{R})$ and some $E,G\in C^{\infty}(\mathbb{T}_{\Theta})_{\geq}$. If $(X_kE)E=E(X_kE)$ and $(X_kG)G=G(X_kG)$ for all $1\leq k\leq 2$, and, moreover,

(a) $(X_2E)(X_1E)=(X_1E)(X_2E)$ and $(X_2G)(X_1G)=(X_1G)(X_2G)$,

(b) $(X_2E)G=G(X_2E)$ and $(X_1G)E=E(X_1G)$, and

(c) $(X_1G)(X_2E)=(X_2E)(X_1G)$,

then

$$
\tau\left(E^{-\frac{1}{2}}g\left(R^{\nabla}(X_1,X_2)X_k,X_k\right)G^{-\frac{1}{2}}\right)=0
$$

for all $1\leq k\leq 2$.

Proof. Pursuant to the commutativity hypotheses and Proposition 2.2.2,

$$
\begin{aligned}
&\tau\left(E^{-\frac{1}{2}}g\left(R^{\nabla}(X_1,X_2)X_1,X_1\right)G^{-\frac{1}{2}}\right)\\
&=\frac{1}{4}\tau\left(E^{-\frac{1}{2}}\left((X_1E)E^{-1}(X_2E)-(X_2E)E^{-1}(X_1E)\right)G^{-\frac{1}{2}}\right)+\\
&\quad\frac{1}{4}\tau\left(E^{-\frac{1}{2}}\left((X_1G)G^{-1}(X_2E)-(X_2E)G^{-1}(X_1G)\right)G^{-\frac{1}{2}}\right)\\
&=\frac{1}{4}\tau\left(E^{-\frac{3}{2}}(X_1E)(X_2E)G^{-\frac{1}{2}}\right)-\frac{1}{4}\tau\left(E^{-\frac{3}{2}}(X_1E)(X_2E)G^{-\frac{1}{2}}\right)+\\
&\quad\frac{1}{4}\tau\left(E^{-\frac{1}{2}}(X_1G)(X_2E)G^{-\frac{3}{2}}\right)-\frac{1}{4}\tau\left(E^{-\frac{1}{2}}(X_1G)(X_2E)G^{-\frac{3}{2}}\right)\\
&=0.
\end{aligned}
$$

Similarly, $\tau\left(E^{-\frac{1}{2}}g\left(R^{\nabla}(X_1,X_2)X_2,X_2\right)G^{-\frac{1}{2}}\right)=0$. $\square$

Remark. If $S = I_2$, i.e. $\left[X_k\right] = \left[\partial_k\right]$, then $(E,G) = (U_1 + U_1^* + 3, U_2 + U_2^* + 3)$ constitutes a nontrivial instance in the sense that G does not commute with E.

Theorem 2.1.3 can now be enhanced.

Corollary 2.2.5. For $n = 2$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[g_{kl}\right] = S^{-1}\mathrm{Diag}(E,E)\left(S^{-1}\right)^t$$

for some $E \in C^\infty(\mathbb{T}_\Theta)_\geq$ and $S \in \mathrm{GL}_2(\mathbb{R})$. Then

$$\tau\left(R_{2112}E^{-1}\right) = 0.$$

Proof. Consider $X_k = {S_k}^l \partial_l$, $1 \leq k \leq 2$. Then

$$[g\left(X_k, X_l\right)] = \left[{S_k}^\alpha g_{\alpha\beta} {S_l}^\beta\right] = S\left(S^{-1}\mathrm{Diag}(E,E)\left(S^{-1}\right)^t\right)S^t = \mathrm{Diag}(E,E).$$

Thus, pursuant to Proposition 2.1.3, and Corollaries 2.2.1 and 2.2.3,

$$\tau\left(g\left(R^\nabla\left(X_i, X_j\right)X_k, X_l\right)E^{-1}\right) = 0$$

for all $1 \leq i,j,k,l \leq 2$. Denote by T the inverse of S. Then

$$\begin{aligned}\tau\left(R_{2112}E^{-1}\right) &= \tau\left(g\left(R^\nabla\left(T_1^i X_i, T_2^j X_j\right)T_1^k X_k, T_2^l X_l\right)E^{-1}\right)\\ &= T_1^i T_2^j T_1^k T_2^l \tau\left(g\left(R^\nabla\left(X_i, X_j\right)X_k, X_l\right)E^{-1}\right) = 0.\end{aligned}$$

$\square$

2.3 4-dimensional Cases

If $\Theta = 0_4$, define $\mathrm{Pf}_{0_4}\left(R_{\alpha\beta\gamma\delta}\right)$ to be

$$\begin{gathered}(R_{1212}R_{3434} - R_{1312}R_{2434} + R_{1412}R_{2334} + R_{2312}R_{1434} - R_{2412}R_{1334} + R_{3412}R_{1234}) + \\ (-R_{1213}R_{3424} + R_{1313}R_{2424} - R_{1413}R_{2324} - R_{2313}R_{1424} + R_{2413}R_{1324} - R_{3413}R_{1224}) + \\ (R_{1214}R_{3423} - R_{1314}R_{2423} + R_{1414}R_{2323} + R_{2314}R_{1423} - R_{2414}R_{1323} + R_{3414}R_{1223}),\end{gathered}$$

then, pursuant to Definition 1.2.23,

$$\mathrm{Pf}(R^{\nabla}) = \mathrm{Pf}_{04}\left(R_{\alpha\beta\gamma\delta}\right)\det\left(\left[g_{kl}\right]\right)^{-1} d\omega_g.$$

Thus, Theorem 1.2.6 is equivalent to

$$\int_{\mathbb{T}^4} \mathrm{Pf}_{04}\left(R_{\alpha\beta\gamma\delta}\right)\det\left(\left[g_{kl}\right]\right)^{-1} d\omega_g = 0,$$

which we anticipate to have some noncommutative adaptations. Nonetheless, as the anterior leviathan formula of $\mathrm{Pf}_{04}\left(R_{\alpha\beta\gamma\delta}\right)$ manifests, the computation involved and, thus, the whole circumstances are extremely complicated and cumbersome. Thence, we shall only exhibit here two palatable instances with a third example consigned to Appendix owing to its unwieldiness. Moreover, in order to minimise the number of commutativity assumptions, we shall define a different generalisation of $\mathrm{Pf}_{04}\left(R_{\alpha\beta\gamma\delta}\right)$ for the instance that resides in Appendix.

For $n = 4$, let g be a Riemannian metric on $C^{\infty}(\mathbb{T}_{\Theta})$. Suppose further that

$$\left[g_{kl}\right] = \mathrm{Diag}(a,b,c,d)$$

for some $a,b,c,d \in C^{\infty}(\mathbb{T}_{\Theta})_{\geq}$. Then, pursuant to the metric hypothesis and Theorem 2.1.2,

$$g(\partial_{i;i},\partial_i) = \frac{1}{2}g_{ii,i}, \qquad g(\partial_{i;i},\partial_k) = -\frac{1}{2}g_{ii,k}, \qquad 1 \leq i,k \leq 4 \text{ with } k \neq i,$$

$$g(\partial_{i;j},\partial_i) = \frac{1}{2}g_{ii,j}, \quad g(\partial_{i;j},\partial_j) = \frac{1}{2}g_{jj,i}, \quad g(\partial_{i;j},\partial_k) = 0, \quad 1 \leq i,j,k \leq 4 \text{ with } i,j,k \text{ distinct.}$$

Thus, since $\{\partial_k\}_{k=1}^{4}$ is an orthogonal basis for $\Gamma^{\infty}(T^{\mathbb{C}}\mathbb{T}_{\Theta})$,

$$\begin{aligned}
\partial_{1;1} &= \frac{1}{2}\left(a_{,1}a^{-1}\partial_1 - a_{,2}b^{-1}\partial_2 - a_{,3}c^{-1}\partial_3 - a_{,4}d^{-1}\partial_4\right),\\
\partial_{2;2} &= \frac{1}{2}\left(b_{,2}b^{-1}\partial_2 - b_{,1}a^{-1}\partial_1 - b_{,3}c^{-1}\partial_3 - b_{,4}d^{-1}\partial_4\right),\\
\partial_{3;3} &= \frac{1}{2}\left(c_{,3}c^{-1}\partial_3 - c_{,1}a^{-1}\partial_1 - c_{,2}b^{-1}\partial_2 - c_{,4}d^{-1}\partial_4\right),\\
\partial_{4;4} &= \frac{1}{2}\left(d_{,4}d^{-1}\partial_4 - d_{,1}a^{-1}\partial_1 - d_{,2}b^{-1}\partial_2 - d_{,3}c^{-1}\partial_3\right),
\end{aligned}$$

$$\begin{aligned}
&\partial_{1;2} = \tfrac{1}{2}\left(a_{,2}a^{-1}\partial_1 + b_{,1}b^{-1}\partial_2\right),\ \partial_{1;3} = \tfrac{1}{2}\left(a_{,3}a^{-1}\partial_1 + c_{,1}c^{-1}\partial_3\right),\ \partial_{1;4} = \tfrac{1}{2}\left(a_{,4}a^{-1}\partial_1 + d_{,1}d^{-1}\partial_4\right),\\
&\partial_{2;1} = \tfrac{1}{2}\left(b_{,1}b^{-1}\partial_2 + a_{,2}a^{-1}\partial_1\right),\ \partial_{2;3} = \tfrac{1}{2}\left(b_{,3}b^{-1}\partial_2 + c_{,2}c^{-1}\partial_3\right),\ \partial_{2;4} = \tfrac{1}{2}\left(b_{,4}b^{-1}\partial_2 + d_{,2}d^{-1}\partial_4\right),\\
&\partial_{3;1} = \tfrac{1}{2}\left(c_{,1}c^{-1}\partial_3 + a_{,3}a^{-1}\partial_1\right),\ \partial_{3;2} = \tfrac{1}{2}\left(c_{,2}c^{-1}\partial_3 + b_{,3}b^{-1}\partial_2\right),\ \partial_{3;4} = \tfrac{1}{2}\left(c_{,4}c^{-1}\partial_3 + d_{,3}d^{-1}\partial_4\right),\\
&\partial_{4;1} = \tfrac{1}{2}\left(d_{,1}d^{-1}\partial_4 + a_{,4}a^{-1}\partial_1\right),\ \partial_{4;2} = \tfrac{1}{2}\left(d_{,2}d^{-1}\partial_4 + b_{,4}b^{-1}\partial_2\right),\ \partial_{4;3} = \tfrac{1}{2}\left(d_{,3}d^{-1}\partial_4 + c_{,4}c^{-1}\partial_3\right).
\end{aligned}$$

Define $\mathrm{Pf}_\Theta\left(R_{\alpha\beta\gamma\delta}\right)$ to be

$$\begin{gathered}(R_{1212}R_{3434} - R_{1312}R_{2434} + R_{1412}R_{2334} + R_{2312}R_{1434} - R_{2412}R_{1334} + R_{3412}R_{1234}) + \\ (-R_{1213}R_{3424} + R_{2424}R_{1313} - R_{1413}R_{2324} - R_{2313}R_{1424} + R_{2413}R_{1324} - R_{3413}R_{1224}) + \\ (R_{1214}R_{3423} - R_{2423}R_{1314} + R_{1414}R_{2323} + R_{2314}R_{1423} - R_{2414}R_{1323} + R_{3414}R_{1223}).\end{gathered}$$

Proposition 2.3.1. For $n = 4$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[g_{kl}\right] = \mathrm{Diag}(a, b, c, d)$$

for some $a \in C^\infty(\mathbb{T}_\Theta)_\geq$ and $b, c, d \in (0, \infty)$. Then

$$\tau\left(\mathrm{Pf}_\Theta\left(R_{\alpha\beta\gamma\delta}\right)(abcd)^{-\frac{1}{2}}\right) = 0.$$

Proof. Pursuant to the metric hypothesis,

$$\begin{aligned}\partial_{1;1} &= \frac{1}{2}\left(a_{,1}a^{-1}\partial_1 - b^{-1}a_{,2}\partial_2 - c^{-1}a_{,3}\partial_3 - d^{-1}a_{,4}\partial_4\right), && \\ \partial_{1;j} &= \frac{1}{2}a_{,j}a^{-1}\partial_1, && 2 \leq j \leq 4, \\ \partial_{i;1} &= \frac{1}{2}a_{,i}a^{-1}\partial_1, && 2 \leq i \leq 4, \\ \partial_{i;j} &= 0, && 2 \leq i, j \leq 4.\end{aligned}$$

Thus,

$$R_{ijkl} = g\left(\left(\partial_{j;l}\right)_{;k}, \partial_i\right) - g\left(\left(\partial_{j;k}\right)_{;l}, \partial_i\right) = 0 - 0 = 0, \qquad 1 \leq i \leq 4 \text{ and } 2 \leq j, k, l \leq 4.$$

Thence,

$$\mathrm{Pf}_\Theta\left(R_{\alpha\beta\gamma\delta}\right) = (0 - 0 + 0 + 0 - 0 + 0) + (-0 + 0 - 0 - 0 + 0 - 0) + (0 - 0 + 0 + 0 - 0 + 0) = 0.$$

Therefore, $\tau\left(\mathrm{Pf}_\Theta\left(R_{\alpha\beta\gamma\delta}\right)(abcd)^{-\frac{1}{2}}\right) = 0.$ □

Proposition 2.3.2. For $n = 4$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[g_{kl}\right] = \mathrm{Diag}(a, a, c, d)$$

for some $a \in C^\infty(\mathbb{T}_\Theta)_\geq$ and $c, d \in (0, \infty)$. If

(a) $a_{,4}a^{-1}a_{,3}{}^2a^{-1}a_{,4} = a_{,4}a^{-1}a_{,4}a_{,3}a^{-1}a_{,3}$,

(b) $a_{,4}a^{-1}a_{,4}a_{,33} = a_{,4}a_{,33}a^{-1}a_{,4}$, and

(c) $a_{,4}a^{-1}a_{,3}a_{,34} = a_{,4}a_{,34}a^{-1}a_{,3}$,

e.g. a commutes with its first and second derivatives, then

$$\tau\left(\mathrm{Pf}_{\Theta}\left(R_{\alpha\beta\gamma\delta}\right)a^{-1}(cd)^{-\frac{1}{2}}\right) = 0.$$

Proof. Pursuant to the metric hypothesis,

$$\partial_{1;1} = \frac{1}{2}\left(a_{,1}a^{-1}\partial_1 - a_{,2}a^{-1}\partial_2 - c^{-1}a_{,3}\partial_3 - d^{-1}a_{,4}\partial_4\right),$$
$$\partial_{2;2} = \frac{1}{2}\left(a_{,2}a^{-1}\partial_2 - a_{,1}a^{-1}\partial_1 - c^{-1}a_{,3}\partial_3 - d^{-1}a_{,4}\partial_4\right),$$
$$\partial_{1;2} = \frac{1}{2}\left(a_{,2}a^{-1}\partial_1 + a_{,1}a^{-1}\partial_2\right), \qquad \partial_{2;1} = \frac{1}{2}\left(a_{,1}a^{-1}\partial_2 + a_{,2}a^{-1}\partial_1\right),$$
$$\partial_{1;j} = \frac{1}{2}a_{,j}a^{-1}\partial_1, \qquad \partial_{2;j} = \frac{1}{2}a_{,j}a^{-1}\partial_2, \qquad 3 \le j \le 4,$$
$$\partial_{i;1} = \frac{1}{2}a_{,i}a^{-1}\partial_1, \qquad \partial_{i;2} = \frac{1}{2}a_{,i}a^{-1}\partial_2, \qquad 3 \le i \le 4,$$
$$\partial_{i;j} = 0, \quad 3 \le i, j \le 4.$$

Thus,

$$\begin{aligned}
R_{ij34} &= g\left(\left(\partial_{j;4}\right)_{;3}, \partial_i\right) - g\left(\left(\partial_{j;3}\right)_{;4}, \partial_i\right) \\
&= 0 - 0 = 0, && 1 \le i \le 4 \text{ and } 3 \le j \le 4, \\
R_{34kl} &= g((\partial_{4;l})_{;k}, \partial_3) - g((\partial_{4;k})_{;l}, \partial_3) \\
&= 0 - \frac{1}{2}(0+0) = 0, && 1 \le k \le 2 \text{ and } 3 \le l \le 4, \\
R_{3412} &= g((\partial_{4;2})_{;1}, \partial_3) - g((\partial_{4;1})_{;2}, \partial_3) \\
&= \frac{1}{2}(0+0) - \frac{1}{2}(0+0) = 0, \\
R_{1j2l} &= g\left(\left(\partial_{j;l}\right)_{;2}, \partial_1\right) - g\left(\left(\partial_{j;2}\right)_{;l}, \partial_1\right) \\
&= 0 - \frac{1}{2}(0+0) = 0, && 3 \le j, l \le 4, \\
R_{1j1l} &= g\left(\left(\partial_{j;l}\right)_{;1}, \partial_1\right) - g\left(\left(\partial_{j;1}\right)_{;l}, \partial_1\right) \\
&= 0 - \frac{1}{2}\left(\left(a_{,j}a^{-1}\right)_{,l}a + \frac{1}{2}a_{,j}a^{-1}a_{,l}\right) = -\frac{1}{2}\left(a_{,jl} - \frac{1}{2}a_{,j}a^{-1}a_{,l}\right), && 3 \le j, l \le 4, \\
R_{2j2l} &= g\left(\left(\partial_{j;l}\right)_{;2}, \partial_2\right) - g\left(\left(\partial_{j;2}\right)_{;l}, \partial_2\right) \\
&= 0 - \frac{1}{2}\left(\left(a_{,j}a^{-1}\right)_{,l}a + \frac{1}{2}a_{,j}a^{-1}a_{,l}\right) = -\frac{1}{2}\left(a_{,jl} - \frac{1}{2}a_{,j}a^{-1}a_{,l}\right), && 3 \le j, l \le 4.
\end{aligned}$$

Thence,

$$
\begin{aligned}
&\mathrm{Pf}_{\Theta}\left(R_{\alpha\beta\gamma\delta}\right)\\
&=(0-0+0+0-0+0)+\\
&\quad \frac{1}{4}\left(-0+\left(a_{,44}-\frac{1}{2}a_{,4}a^{-1}a_{,4}\right)\left(a_{,33}-\frac{1}{2}a_{,3}a^{-1}a_{,3}\right)-\right.\\
&\qquad \left.\left(a_{,43}-\frac{1}{2}a_{,4}a^{-1}a_{,3}\right)\left(a_{,34}-\frac{1}{2}a_{,3}a^{-1}a_{,4}\right)-0+0-0\right)+\\
&\quad \frac{1}{4}\left(0-\left(a_{,43}-\frac{1}{2}a_{,4}a^{-1}a_{,3}\right)\left(a_{,34}-\frac{1}{2}a_{,3}a^{-1}a_{,4}\right)+\right.\\
&\qquad \left.\left(a_{,44}-\frac{1}{2}a_{,4}a^{-1}a_{,4}\right)\left(a_{,33}-\frac{1}{2}a_{,3}a^{-1}a_{,3}\right)+0-0+0\right)\\
&=\frac{1}{4}\left(\left(a_{,44}a_{,33}-\frac{1}{2}a_{,4}a^{-1}a_{,4}a_{,33}-\frac{1}{2}a_{,44}a_{,3}a^{-1}a_{,3}+\frac{1}{4}a_{,4}a^{-1}a_{,4}a_{,3}a^{-1}a_{,3}\right)-\right.\\
&\qquad \left.\left(a_{,43}a_{,34}-\frac{1}{2}a_{,4}a^{-1}a_{,3}a_{,34}-\frac{1}{2}a_{,43}a_{,3}a^{-1}a_{,4}+\frac{1}{4}a_{,4}a^{-1}{a_{,3}}^{2}a^{-1}a_{,4}\right)\right)+\\
&\quad \frac{1}{4}\left(-\left(a_{,43}a_{,34}-\frac{1}{2}a_{,4}a^{-1}a_{,3}a_{,34}-\frac{1}{2}a_{,43}a_{,3}a^{-1}a_{,4}+\frac{1}{4}a_{,4}a^{-1}{a_{,3}}^{2}a^{-1}a_{,4}\right)+\right.\\
&\qquad \left.\left(a_{,44}a_{,33}-\frac{1}{2}a_{,4}a^{-1}a_{,4}a_{,33}-\frac{1}{2}a_{,44}a_{,3}a^{-1}a_{,3}+\frac{1}{4}a_{,4}a^{-1}a_{,4}a_{,3}a^{-1}a_{,3}\right)\right)\\
&=\frac{1}{4}\left(a_{,44}a_{,33}+a_{,4}a_{,334}-a_{,4}a_{,33}a^{-1}a_{,4}\right)+\frac{1}{4}\left(a_{,443}a_{,3}+a_{,44}a_{,33}-a_{,44}a_{,3}a^{-1}a_{,3}\right)-\\
&\quad \frac{1}{4}\left(a_{,43}a_{,34}+a_{,4}a_{,343}-a_{,4}a_{,34}a^{-1}a_{,3}\right)-\frac{1}{4}\left(a_{,434}a_{,3}+a_{,43}a_{,34}-a_{,43}a_{,3}a^{-1}a_{,4}\right).
\end{aligned}
$$

Therefore, since τ is invariant under the group action generated by $\{\partial_k\}_{k=1}^{4}$,

$$
\begin{aligned}
&\tau\left(\mathrm{Pf}_{\Theta}\left(R_{\alpha\beta\gamma\delta}\right)a^{-1}(cd)^{-\frac{1}{2}}\right)\\
&=\frac{1}{4}(cd)^{-\frac{1}{2}}\tau\left(\left(a_{,4}a_{,33}a^{-1}\right)_{,4}\right)-\frac{1}{4}(cd)^{-\frac{1}{2}}\tau\left(\left(a_{,43}a_{,3}a^{-1}\right)_{,4}\right)+\\
&\quad \frac{1}{4}(cd)^{-\frac{1}{2}}\tau\left(\left(a_{,44}a_{,3}a^{-1}\right)_{,3}\right)-\frac{1}{4}(cd)^{-\frac{1}{2}}\tau\left(\left(a_{,4}a_{,34}a^{-1}\right)_{,3}\right)\\
&=0-0+0-0\\
&=0.
\end{aligned}
$$

□

Chapter 3

Riemannian Geometry on Quantum Discs and 2-Spheres

3.1 Levi-Civita Connections

We shall imitate the method delineated in Section 2.1 to transcribe Riemannian geometry for quantum discs and 2-spheres. In particular, the results and proofs exhibited here are adaptations of those presented in [19] and [16] for noncommutative tori.

With a view to proceeding and attaining results concretely, we shall identify $C\left(\overline{\mathbb{D}}_q\right)$ with $\gamma\left(C\left(\overline{\mathbb{D}}_q\right)\right)$ in this chapter (Proposition 1.3.3).

Recall in geometry that there are two equivalent definitions for $\overline{\mathbb{D}}$, videlicet,

$$\overline{\mathbb{D}}=\left\{(x,y)\in\mathbb{R}^2:\|(x,y)\|\leq 1\right\} \text{ and } \overline{\mathbb{D}}=\{z\in\mathbb{C}:|z|\leq 1\}.$$

Especially, the two definitions are yoked to each other via

$$x=\frac{1}{2}(z+\overline{z}) \text{ and } y=\frac{1}{2i}(z-\overline{z}).$$

For $C\left(\overline{\mathbb{D}}_q\right)$, since $\sigma_{C\left(\overline{\mathbb{D}}_q\right)}\left(z_q z_q^*\right)$ coincides with $\left\{1-q^k\right\}_{k\in\mathbb{N}}\cup\{1\}$, z_q could be deemed the noncommutative counterpart of z. Thus, the foregoing coupling suggests that we could deem

$$\frac{1}{2}\left(z_q+z_q^*\right) \text{ and } \frac{1}{2i}\left(z_q-z_q^*\right),$$

respectively, the counterparts of x and y.

For lucidity and convenience, denote $\frac{1}{2}\left(z_q+z_q^*\right)$ and $\frac{1}{2i}\left(z_q-z_q^*\right)$, respectively, by x_q and y_q. Moreover, denote $\sum\limits_{k\in\mathbb{N}} q^{k-1}e_{kk}$ by Q_q. Naturally, x_q and y_q are self-adjoint, and Q_q is an element of $\mathcal{K}\left(\mathcal{H}_{\text{sep}}\right)_{\text{sa}}$ that satisfies $\lim\limits_{q\to 1^-} Q_q=1$ strongly.

Lemma 3.1.1. $\left[x_q, y_q\right] = \dfrac{1-q}{2i} Q_q$.

Proof.

$$\begin{aligned}
\left[x_q, y_q\right] &= \left[\frac{1}{2}\left(z_q + z_q^*\right), \frac{1}{2i}\left(z_q - z_q^*\right)\right] = \frac{1}{4i}\left(\left[z_q, z_q\right] + \left[z_q^*, z_q\right] - \left[z_q, z_q^*\right] - \left[z_q^*, z_q^*\right]\right) \\
&= \frac{1}{2i}\left[z_q^*, z_q\right] \\
&= \frac{1}{2i}\left(\sum_{k \in \mathbb{N}}\left(1-q^k\right) e_{kk} - \sum_{k \in \mathbb{N}}\left(1-q^{k-1}\right) e_{kk}\right) = \frac{1}{2i}\sum_{k \in \mathbb{N}} q^{k-1}(1-q) e_{kk} \\
&= \frac{1-q}{2i} Q_q.
\end{aligned}$$

□

Lemma 3.1.2.

$$\partial_1 := -\frac{1}{i}\frac{2}{1-q}[\cdot, y_q] \text{ and } \partial_2 := \frac{1}{i}\frac{2}{1-q}[\cdot, x_q]$$

are *-derivations on $C\left(\overline{\mathbb{D}}_q\right)$. Moreover,

$$\partial_1 x_q = Q_q, \quad \partial_1 y_q = 0,$$
$$\partial_2 x_q = 0, \quad \partial_2 y_q = Q_q.$$

In particular,

$$\operatorname{Im}(\partial_k) \subseteq \mathcal{K}\left(\mathcal{H}_{\text{sep}}\right) \text{ for all } 1 \leq k \leq 2, \text{ and}$$
$$\lim_{q \to 1^-} \partial_1 x_q = 1 \text{ strongly and } \lim_{q \to 1^-} \partial_2 y_q = 1 \text{ strongly.}$$

Proof. By virtue of the Jacobi identity of $[\cdot,\cdot]$, ∂_1 and ∂_2 are derivations on $C\left(\overline{\mathbb{D}}_q\right)$. Thus, it is sufficient to verify that $\partial_k \circ^* = ^* \circ \partial_k$ for all $1 \leq k \leq 2$. Let $a \in C\left(\overline{\mathbb{D}}_q\right)$. Then
$\partial_1 a^* = -\frac{1}{i}\frac{2}{1-q}\left[a^*, y_q\right] = -\frac{1}{i}\frac{2}{1-q}\left[y_q^*, a\right]^* = -\frac{1}{i}\frac{2}{1-q}\left[y_q, a\right]^* = \left(-\frac{1}{i}\frac{2}{1-q}\left[a, y_q\right]\right)^* = (\partial_1 a)^*$.
Similarly, $\partial_2 a^* = (\partial_2 a)^*$.
Pursuant to Lemma 3.1.1, $\partial_1 x_q = -\frac{1}{i}\frac{2}{1-q}\left[x_q, y_q\right] = Q$.
Similarly, $\partial_2 y_q = Q$.
On the other hand, $\partial_1 y_q = -\frac{1}{i}\frac{2}{1-q}\left[y_q, y_q\right] = 0$.
Similarly, $\partial_2 x_q = 0$. □

Remark. This suggests the notion that ∂_1 and ∂_2, respectively, be identified with the "noncommutative coordinate vector fields" corresponding to x_q and y_q. In particular, if $\hbar = \frac{1-q}{2}$, then

$$\lim_{\hbar \to 0^+} -\frac{1}{i\hbar}\left[x_q, y_q\right] = \{x, y\},$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket on $C^\infty(\mathbb{R}^2)$. Thus, $[\cdot,\cdot]$ could be viewed as a Moyal bracket obtained from $\{\cdot,\cdot\}$ via deformation quantisation (p.412 of [17]).

Denote by $\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$ the free left $C\left(\overline{\mathbb{D}}_q\right)$-module spanned by $\{\partial_1, \partial_2\}$, which will act, like $\Gamma^\infty\left(T^{\mathbb{C}}\mathbb{T}_\Theta\right)$, as the "complexified tangent bundle" of $C\left(\overline{\mathbb{D}}_q\right)$.

Lemma 3.1.3. $[\partial_1, \partial_2] = \operatorname{ad}\left(i\frac{2}{1-q}Q_q\right)$. In particular,

$$\lim_{q \to 1^-} [\partial_1, \partial_2](a) = 0 \text{ strongly for all } a \in C\left(\overline{\mathbb{D}}_q\right).$$

Proof.

$$\begin{aligned}
[\partial_1, \partial_2] &= -\frac{1}{i}\frac{2}{1-q}\left[\frac{1}{i}\frac{2}{1-q}\left[\cdot, x_q\right], y_q\right] - \frac{1}{i}\frac{2}{1-q}\left[-\frac{1}{i}\frac{2}{1-q}\left[\cdot, y_q\right], x_q\right] \\
&= \left(\frac{2}{1-q}\right)^2 \left(\left[\left[\cdot, x_q\right], y_q\right] - \left[\left[\cdot, y_q\right], x_q\right]\right) = \left(\frac{2}{1-q}\right)^2 \left(\left[\left[\cdot, x_q\right], y_q\right] + \left[\left[y_q, \cdot\right], x_q\right]\right) \\
&= \left(\frac{2}{1-q}\right)^2 \left(-\left[\left[x_q, y_q\right], \cdot\right]\right) && \text{(by the Jacobi identity of } [\cdot,\cdot]) \\
&= \left(\frac{2}{1-q}\right)^2 \left(-\left[\frac{1-q}{2i}Q_q, \cdot\right]\right) = \left[i\frac{2}{1-q}Q_q, \cdot\right] && \text{(by Lemma 3.1.1)} \\
&= \operatorname{ad}\left(i\frac{2}{1-q}Q_q\right).
\end{aligned}$$

□

Lemma 3.1.4. Let $K, K_1, K_2 \in i\mathcal{K}\left(\mathcal{H}_{\text{sep}}\right)_{\text{sa}}$. Then

1. $\operatorname{ad}(K)$ forms a *-derivation on $C\left(\overline{\mathbb{D}}_q\right)$, and
2. $[\operatorname{ad}(K_1), \operatorname{ad}(K_2)]$ is an element of $\operatorname{ad}\left(i\mathcal{K}\left(\mathcal{H}_{\text{sep}}\right)_{\text{sa}}\right)$.

Proof. By virtue of the Jacobi identity of $[\cdot,\cdot]$, $\operatorname{ad}(K)$ forms derivations on $C\left(\overline{\mathbb{D}}_q\right)$. Thus, it is sufficient to verify that $\operatorname{ad}(K)\circ^* =^* \circ\operatorname{ad}(K)$. Let $a \in C\left(\overline{\mathbb{D}}_q\right)$. Then

$\operatorname{ad}(K)(a^*) = [K, a^*] = [a, K^*]^* = [a, -K]^* = [K, a]^* = \operatorname{ad}(K)(a)^*$.

By virtue of the Jacobi identity of $[\cdot,\cdot]$,

$[\operatorname{ad}(K_1), \operatorname{ad}(K_2)] = [K_1, [K_2, \cdot]] - [K_2, [K_1, \cdot]] = [K_1, [K_2, \cdot]] + [K_2, [\cdot, K_1]]$

$$= -[\cdot,[K_1,K_2]] = [[K_1,K_2],\cdot]$$

$$= \mathrm{ad}([K_1,K_2]).$$

Thus, it is sufficient to verify that $[K_1,K_2]^* = -[K_1,K_2]$. Notwithstanding,

$[K_1,K_2]^* = [K_2^*,K_1^*] = [-K_2,-K_1] = -[K_1,K_2]$. $\square$

Proposition 3.1.1. Equipped with the commutator bracket, $\mathrm{Span}_{\mathbb{R}}(\{\partial_1,\partial_2\}) \oplus \mathrm{ad}\left(i\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)_{\mathrm{sa}}\right)$ forms a (real) Lie algebra and will be denoted by $\mathfrak{D}(\overline{\mathbb{D}}_q)$.

Proof. Pursuant to Lemmas 3.1.3 and 3.1.4, it is sufficient to verify that $[\partial_k,\mathrm{ad}(K)]$ belongs to $\mathrm{ad}\left(i\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)_{\mathrm{sa}}\right)$ for all $1 \leq k \leq 2$ and $K \in i\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)_{\mathrm{sa}}$. By virtue of the Jacobi identity of $[\cdot,\cdot]$,

$$\begin{aligned}
[\partial_1,\mathrm{ad}(K)] &= -\frac{1}{i}\frac{2}{1-q}\left[[K,\cdot],y_q\right] - \left[K, -\frac{1}{i}\frac{2}{1-q}\left[\cdot,y_q\right]\right] \\
&= -\frac{1}{i}\frac{2}{1-q}\left(\left[[K,\cdot],y_q\right] - \left[K,\left[\cdot,y_q\right]\right]\right) = -\frac{1}{i}\frac{2}{1-q}\left(\left[[K,\cdot],y_q\right] + \left[\left[\cdot,y_q\right],K\right]\right) \\
&= -\frac{1}{i}\frac{2}{1-q}\left(-\left[\left[y_q,K\right],\cdot\right]\right) = [\partial_1 K,\cdot] \\
&= \mathrm{ad}(\partial_1 K).
\end{aligned}$$

Thus, pursuant to Lemma 3.1.2, it it sufficient to verify that $(\partial_1 K)^* = -\partial_1 K$. Notwithstanding, $(\partial_1 K)^* = \partial_1 K^* = \partial_1(-K) = -\partial_1 K$.

Similarly, $[\partial_2,\mathrm{ad}(K)]$ belongs to $\mathrm{ad}\left(i\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)_{\mathrm{sa}}\right)$. $\square$

We are now primed to regenerate Riemannian geometry on $C\left(\overline{\mathbb{D}}_q\right)$.

Definition 3.1.1. Let g be an $\mathbb{R}$-bilinear form on $\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$. Then g is called a Riemannian metric on $C\left(\overline{\mathbb{D}}_q\right)$ if g is a $C\left(\overline{\mathbb{D}}_q\right)$-valued inner product on $\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$, videlicet,

(a) $g(\cdot,X)$ is $C\left(\overline{\mathbb{D}}_q\right)$-linear for all $X \in \Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$,

(b) $g(Y,X) = g(X,Y)^*$ for all $X,Y \in \Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$,

(c) $\left[g(\partial_k,\partial_l)\right]$ is an invertible and positive element of $\mathrm{M}_2\left(C\left(\overline{\mathbb{D}}_q\right)\right)$, and

(d) $g(\partial_k,\partial_l)$ is self-adjoint for all $1 \leq k,l \leq 2$.

Definition 3.1.2. Let D be an $\mathbb{R}$-bilinear map from $\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right) \times \mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$ to $\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$. Then D is called a (linear) connection on $C\left(\overline{\mathbb{D}}_q\right)$ if

(a) $D(aX,Y) = (Ya)X + aD(X,Y)$ for all $X \in \Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$, $Y \in \mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$, and $a \in C\left(\overline{\mathbb{D}}_q\right)$,

(b) $D(X, \mathrm{ad}(K)) = KX$ for all $X \in \Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$ and $K \in i\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)_{\mathrm{sa}}$, and

(c) $g(D(\partial_k, \partial_l), \partial_m)$ is self-adjoint for all $1 \le k, l, m \le 2$.

In accordance with notational convention, $D(X,Y)$ will be denoted by $D_Y X$.

Remark. By virtue of the injectivity of $\mathrm{ad}\Big|_{i\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)_{\mathrm{sa}}}$, condition (b) is well-defined.

Definition 3.1.3. Let D be a connection on $C\left(\overline{\mathbb{D}}_q\right)$. Define
$R^D : \mathfrak{D}\left(\overline{\mathbb{D}}_q\right) \times \mathfrak{D}\left(\overline{\mathbb{D}}_q\right) \longrightarrow \mathrm{End}_{\mathbb{C}}\left(\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)\right)$ by

$$R^D(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]} \text{ for } X, Y \in \mathfrak{D}\left(\overline{\mathbb{D}}_q\right).$$

Then R^D is called the curvature of D.

Remark. Naturally, R^D is $\mathbb{R}$-bilinear.

Proposition 3.1.2. Let D be a connection on $C\left(\overline{\mathbb{D}}_q\right)$. Then $R^D(X,Y)$ belongs to
$\mathrm{End}_{C\left(\overline{\mathbb{D}}_q\right)}\left(\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)\right)$ for all $X, Y \in \mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$.

Proof. Let $Z \in \Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$ and $a \in C\left(\overline{\mathbb{D}}_q\right)$. Then

$$\begin{aligned}
R^D(X,Y)(aZ) &= D_X D_Y(aZ) - D_Y D_X(aZ) - D_{[X,Y]}(aZ) \\
&= (((XYa)Z + (Ya)D_X Z) + ((Xa)D_Y Z + aD_X D_Y Z)) - \\
&\quad (((YXa)Z + (Xa)D_Y Z) + ((Ya)D_X Z + aD_Y D_X Z)) - ((XYa - YXa)Z + aD_{[X,Y]}Z) \\
&= a(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z) \\
&= aR^D(X,Y)(Z).
\end{aligned}$$

□

Proposition 3.1.3. Let D be a connection on $C\left(\overline{\mathbb{D}}_q\right)$. Then $R^D(\mathrm{ad}(K), \cdot) \equiv 0$ for all $K \in i\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)_{\mathrm{sa}}$.

Proof. Let $X \in \mathfrak{D}(\overline{\mathbb{D}}_q)$ and $a \in C\left(\overline{\mathbb{D}}_q\right)$. Then

$$\begin{aligned}
[\mathrm{ad}(K), X]a &= [K, Xa] - X[K, a] \\
&= (K(Xa) - (Xa)K) - (((XK)a + K(Xa)) - ((Xa)K + a(XK)))
\end{aligned}$$

$$= (-XK)a - a(-XK)$$

$$= \mathrm{ad}(-XK)(a).$$

Thus, for every $Z \in \Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$,

$$\begin{aligned} R^D(\mathrm{ad}(K), X)(Z) &= D_{\mathrm{ad}(K)} D_X Z - D_X D_{\mathrm{ad}(K)} Z - D_{[\mathrm{ad}(K), X]} Z \\ &= D_{\mathrm{ad}(K)} D_X Z - D_X D_{\mathrm{ad}(K)} Z - D_{\mathrm{ad}(-XK)} Z \\ &= K D_X Z - ((XK)Z + K D_X Z) - (-XK)Z \\ &= 0. \end{aligned}$$

□

Definition 3.1.4. Let g and D, respectively, be a Riemannian metric and a connection on $C\left(\overline{\mathbb{D}}_q\right)$. Then D is said to be g-compatible if $Z(g(X,Y)) = g(D_Z X, Y) + g(X, D_Z Y)$ for all $X, Y \in \Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$ and $Z \in \mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$.

Definition 3.1.5. Let D be a connection on $C\left(\overline{\mathbb{D}}_q\right)$. Then D is said to be torsion-free if $D_{\partial_l}\partial_k - D_{\partial_k}\partial_l = 0$ for all $1 \le k, l \le 2$.

Remark. Since $[\partial_1, \partial_2]$ does not belong to $\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$ and $\lim\limits_{q \to 1^-} [\partial_1, \partial_2](a) = 0$ strongly, we ignore $[\partial_1, \partial_2]$ here though it does not vanish.

Proposition 3.1.4. Let g be a Riemannian metric on $C\left(\overline{\mathbb{D}}_q\right)$. Then there exists a unique g-compatible and torsion-free connection ∇ on $\left(C\left(\overline{\mathbb{D}}_q\right), g\right)$, called the Levi-Civita connection. In particular, it is completely determined via

$$g\left(\nabla_{\partial_j}\partial_i, \partial_k\right) = \frac{1}{2}\left(\partial_j g(\partial_i, \partial_k) + \partial_i g\left(\partial_j, \partial_k\right) - \partial_k g\left(\partial_j, \partial_i\right)\right) \text{ for all } 1 \le i, j, k \le 2.$$

Proof.

Uniqueness:

Let D be a g-compatible and torsion-free connection on $C\left(\overline{\mathbb{D}}_q\right)$. By virtue of the g-compatibility,

$$\partial_j g(\partial_i, \partial_k) = g\left(D_{\partial_j}\partial_i, \partial_k\right) + g\left(\partial_i, D_{\partial_j}\partial_k\right), \qquad 1 \le i, j, k \le 2,$$

$$\partial_i g\left(\partial_k, \partial_j\right) = g\left(D_{\partial_i}\partial_k, \partial_j\right) + g\left(\partial_k, D_{\partial_i}\partial_j\right), \qquad 1 \le i, j, k \le 2,$$

$$\partial_k g\left(\partial_j,\partial_i\right)=g\left(D_{\partial_k}\partial_j,\partial_i\right)+g\left(\partial_j,D_{\partial_k}\partial_i\right), \qquad 1\le i,j,k\le 2.$$

Then, by virtue of the torsion-freeness and condition (c) in Definition 3.1.2, we obtain, via rearranging the equations,

$$\begin{aligned} g\left(D_{\partial_j}\partial_i,\partial_k\right) &= \partial_j g(\partial_i,\partial_k)-g\left(D_{\partial_k}\partial_j,\partial_i\right) && 1\le i,j,k\le 2,\\ g\left(D_{\partial_j}\partial_i,\partial_k\right) &= \partial_i g\left(\partial_k,\partial_j\right)-g\left(\partial_j,D_{\partial_k}\partial_i\right) && 1\le i,j,k\le 2,\\ 0 &= -\partial_k g\left(\partial_j,\partial_i\right)+g\left(D_{\partial_k}\partial_j,\partial_i\right)+g\left(\partial_j,D_{\partial_k}\partial_i\right) && 1\le i,j,k\le 2. \end{aligned}$$

Thus, we obtain, via adding the three equations together,

$$2g\left(D_{\partial_j}\partial_i,\partial_k\right)=\partial_j g(\partial_i,\partial_k)+\partial_i g\left(\partial_k,\partial_j\right)-\partial_k g\left(\partial_j,\partial_i\right), \quad 1\le i,j,k\le 2,$$

which is exactly the determining formula of ∇.

Existence:

By virtue of the invertibility of $\left[g(\partial_k,\partial_l)\right]$, for every $1\le i,j\le 2$, $\left\{g\left(\nabla_{\partial_j}\partial_i,\partial_k\right)\right\}_{k=1}^{2}$ completely determines $\nabla_{\partial_j}\partial_i$. Thus, ∇ can be extended from $\{\partial_1,\partial_2\}\times\{\partial_1,\partial_2\}$ to $\{\partial_1,\partial_2\}\times\mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$ via

$$\nabla_{\sum\limits_{l=1}^{2}\lambda_l\partial_l+\mathrm{ad}(K)}\partial_k=\sum_{l=1}^{2}\lambda_l\nabla_{\partial_l}\partial_k+K\partial_k \text{ for } 1\le k\le 2,\, K\in i\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)_{\mathrm{sa}}, \text{ and } \lambda_1,\lambda_2\in\mathbb{R}.$$

Thence, ∇ can be extended from $\{\partial_1,\partial_2\}\times\mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$ to $\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)\times\mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$ via

$$\nabla_Y\left(\sum_{k=1}^{2}a_k\partial_k\right)=\sum_{k=1}^{2}\left(\left(\partial_Y a_k\right)\partial_k+a_k\nabla_Y\partial_k\right) \text{ for } Y\in\mathfrak{D}\left(\overline{\mathbb{D}}_q\right) \text{ and } a_1,a_2\in C\left(\overline{\mathbb{D}}_q\right).$$

By construction, ∇ is $\mathbb{R}$-bilinear and satisfies conditions (b) and (c) in Definition 3.1.2. Therefore, it is sufficient to verify that it satisfies condition (a) in Definition 3.1.2, and is g-compatible and torsion-free.

To substantiate condition (a), by virtue of the $\mathbb{R}$-bilinearity, it is sufficient to verify that $\nabla_Y(ab\partial_k)=(Ya)(b\partial_k)+a(\nabla_Y(b\partial_k))$ for all $1\le k\le 2$, $Y\in\mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$, and $a,b\in C\left(\overline{\mathbb{D}}_q\right)$. Notwithstanding,

$$\begin{aligned} \nabla_Y(ab\partial_k) &= (Y(ab))\partial_k+ab\nabla_Y\partial_k\\ &= ((Ya)b+a(Yb))\partial_k+ab\nabla_Y\partial_k\\ &= (Ya)(b\partial_k)+a((Yb)\partial_k+b\nabla_Y\partial_k)\\ &= (Ya)(b\partial_k)+a(\nabla_Y(b\partial_k)). \end{aligned}$$

To substantiate the g-compatibility, by virtue of conditions (a) and (b) in Definition 3.1.1, it is sufficient to verify that $g\left(\nabla_{\partial_k}\partial_i,\partial_j\right)+g\left(\partial_i,\nabla_{\partial_k}\partial_j\right)=\partial_k g\left(\partial_i,\partial_j\right)$ for all $1\leq i,j,k\leq 2$. Notwithstanding,

$$
\begin{aligned}
& g\left(\nabla_{\partial_k}\partial_i,\partial_j\right)+g\left(\partial_i,\nabla_{\partial_k}\partial_j\right) \\
& =g\left(\nabla_{\partial_k}\partial_i,\partial_j\right)+g\left(\nabla_{\partial_k}\partial_j,\partial_i\right) \\
& =\frac{1}{2}\left(\partial_k g\left(\partial_i,\partial_j\right)+\partial_i g\left(\partial_k,\partial_j\right)-\partial_j g(\partial_k,\partial_i)\right)+\frac{1}{2}\left(\partial_k g\left(\partial_j,\partial_i\right)+\partial_j g(\partial_k,\partial_i)-\partial_i g\left(\partial_k,\partial_j\right)\right) \\
& =\frac{1}{2}\left(\partial_k g\left(\partial_i,\partial_j\right)+\partial_k g\left(\partial_j,\partial_i\right)\right) \\
& =\partial_k g\left(\partial_i,\partial_j\right).
\end{aligned}
$$

Finally, ∇ is torsion-free because its determining formula is symmetric in the indices i and j. □

Proposition 3.1.5. Let g be a Riemannian metric on $C\left(\overline{\mathbb{D}}_q\right)$.

1. $R^\nabla(X,Y)Z+R^\nabla(Y,Z)X+R^\nabla(Z,X)Y=0$ for all $X,Y,Z\in\mathrm{Span}_{\mathbb{R}}(\{\partial_1,\partial_2\})$.
2. $R^\nabla(Y,X)=-R^\nabla(X,Y)$ for all $X,Y\in\mathfrak{D}\left(\overline{\mathbb{D}}_q\right)$.

Proof. It is sufficient to verify that

$$R^\nabla\left(\partial_i,\partial_j\right)\partial_k+R^\nabla\left(\partial_j,\partial_k\right)\partial_i+R^\nabla\left(\partial_k,\partial_i\right)\partial_j=0$$

for all $1\leq i,j,k\leq 2$. By virtue of the torsion-freeness,

$$
\begin{aligned}
& R^\nabla\left(\partial_i,\partial_j\right)\partial_k+R^\nabla\left(\partial_j,\partial_k\right)\partial_i+R^\nabla\left(\partial_k,\partial_i\right)\partial_j \\
& =\left(\nabla_{\partial_i}\nabla_{\partial_j}\partial_k-\nabla_{\partial_j}\nabla_{\partial_i}\partial_k\right)+\left(\nabla_{\partial_j}\nabla_{\partial_k}\partial_i-\nabla_{\partial_k}\nabla_{\partial_j}\partial_i\right)+\left(\nabla_{\partial_k}\nabla_{\partial_i}\partial_j-\nabla_{\partial_i}\nabla_{\partial_k}\partial_j\right) \\
& =\nabla_{\partial_i}\left(\nabla_{\partial_j}\partial_k-\nabla_{\partial_k}\partial_j\right)+\nabla_{\partial_j}\left(\nabla_{\partial_k}\partial_i-\nabla_{\partial_i}\partial_k\right)+\nabla_{\partial_k}\left(\nabla_{\partial_i}\partial_j-\nabla_{\partial_j}\partial_i\right) \\
& =\nabla_{\partial_i}0+\nabla_{\partial_j}0+\nabla_{\partial_k}0 \\
& =0.
\end{aligned}
$$

Listing 2 results directly from Definition 3.1.3. □

For lucidity and convenience, the notational conventions adopted in Section 2.1 will also be adopted in the sequel.

3.2 Total Curvatures of Quantum 2-Spheres

Recall in algebraic topology the Hairy Ball Theorem (p.435 of [13]), videlicet,

there exists a nonvanishing element in $\Gamma(T\mathbb{S}^n)$ if and only if n is odd,

which implies that every element of $\mathfrak{X}(\mathbb{S}^2)$ must vanish somewhere and, thus, it is impossible to endow $\mathbb{S}^2$ with a flat Riemannian metric (p.145 of [10]). Thence, the metric induced from the standard Euclidean one on $\mathbb{R}^3$ would be the next most natural choice for a Riemannian metric on $\mathbb{S}^2$.

In both charts of the smooth atlas $\left\{y^2_{\pm}\right\}$ (Example 1.2.1), this induced metric is completely determined via

$$\begin{bmatrix} g(\partial_x,\partial_x)(x,y) & g(\partial_x,\partial_y)(x,y) \\ g(\partial_y,\partial_x)(x,y) & g(\partial_y,\partial_y)(x,y) \end{bmatrix} = \frac{4}{(1+x^2+y^2)^2}\mathrm{Diag}(1,1) \text{ for all } (x,y)\in\mathbb{R}^2$$

(p.25 of [10]). Moreover, the inverse images of $\overline{\mathbb{D}}$ under y^2_+ and y^2_-, respectively, coincide with the closed upper and lower hemispheres of $\mathbb{S}^2$. Thus, in accordance with the remark that succeeds Definition 1.3.9,

$$4\left(1+x_q^2+y_q^2\right)^{-2}\mathrm{Diag}(1,1)$$

could be deemed the analogue of this metric for Podleś spheres and quantum complex projective lines.

Unfortunately, we could not establish results similar to those in Section 2.2 owing to some technical difficulties, e.g. the lack of flat metrics on spheres. Nonetheless, an asymptotic behaviour of the total curvatures of quantum complex projective lines has been found along the course of our study on quantum discs and 2-spheres.

Lemma 3.2.1. $4\left(1+x_q^2+y_q^2\right)^{-2} = \sum_{k\in\mathbb{N}} \frac{16}{\left(4-q^{k-1}-q^k\right)^2}e_{kk}$. In particular, $4\left(1+x_q^2+y_q^2\right)^{-2}\mathrm{Diag}(1,1)$ is an invertible and positive element of $\mathrm{M}_2\left(C\left(\overline{\mathbb{D}}_q\right)\right)$.

Proof. It is sufficient to verify that $\frac{1}{4}\left(1+x_q^2+y_q^2\right)^2 = \sum_{k\in\mathbb{N}} \frac{\left(4-q^{k-1}-q^k\right)^2}{16} e_{kk}$. Notwithstanding,

$$\begin{aligned}
&\frac{1}{4}\left(1+x_q^2+y_q^2\right)^2 \\
&= \frac{1}{4}\left(1+\left(\frac{1}{2}\left(z_q+z_q^*\right)\right)^2+\left(\frac{1}{2i}\left(z_q-z_q^*\right)\right)^2\right)^2 = \frac{1}{16}\left(2+z_q^*z_q+z_qz_q^*\right)^2 \\
&= \frac{1}{16}\left(2+\sum_{k\in\mathbb{N}}\left(1-q^k\right)e_{kk}+\sum_{k\in\mathbb{N}}\left(1-q^{k-1}\right)e_{kk}\right)^2 = \frac{1}{16}\left(\sum_{k\in\mathbb{N}}\left(4-q^{k-1}-q^k\right)e_{kk}\right)^2 \\
&= \frac{1}{16}\sum_{k\in\mathbb{N}}\left(4-q^{k-1}-q^k\right)^2 e_{kk} = \sum_{k\in\mathbb{N}} \frac{\left(4-q^{k-1}-q^k\right)^2}{16} e_{kk}.
\end{aligned}$$

□

For lucidity and convenience,

$$E_q := 4\left(1+x_q^2+y_q^2\right)^{-2}.$$

Moreover, define E_q^0 to be 1 and, for every $k\in\mathbb{N}$, let

$$E_q^k := \frac{16}{\left(4-q^{k-1}-q^k\right)^2}.$$

Naturally, $E_q = \sum_{k\in\mathbb{N}} E_q^k e_{kk}$.

By virtue of Lemma 3.2.1, define a Riemannian metric on $C\left(\overline{\mathbb{D}}_q\right)$, viewed as a hemisphere of $C\left(\mathbb{S}^2_{qq}\right)$ or $C\left(\mathbb{CP}^1_{qq}\right)$, via

$$\left[g_{ij}\right] = E_q \mathrm{Diag}\left(1,1\right).$$

Unlike the circumstances for noncommutative tori, there is no canonical choice in $\mathcal{B}\left(C\left(\overline{\mathbb{D}}_q\right),\mathbb{C}\right)$ that acts as a "noncommutative Borel or Lebesgue integration over $\mathbb{D}$". Thus, to obtain an analogue on $C\left(\overline{\mathbb{D}}_q\right)$ of the integration against the Riemannian volume form, consider an $A\in C\left([0,1),\mathcal{K}\left(\mathcal{H}_{\mathrm{sep}}\right)\right)$ that satisfies

(a) $A(q) = \sum_{k\in\mathbb{N}} f(q)_k e_{kk}$ for some $\{f(q)_k\}_{k\in\mathbb{N}} \subseteq C([0,1),(0,\infty))$,

(b) $\mathrm{Tr}\left(A(q)E_q\right) = 2\pi$ for all $q\in[0,1)$, and

(c) there exist an $\varepsilon>0$ and a $\{g_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ such that $\sum_{k\in\mathbb{N}} g_k < \infty$ and $kf(q)_k \le g_k$ for all $k\in\mathbb{N}$ and $q\in(1-\varepsilon,1)$.

For instance,

$$A(q)=\frac{180}{\pi^3}\left(\sum_{k\in\mathbb{N}}k^{-4}e_{kk}\right)E_q^{-1}\text{ or }A(q)=\frac{2\pi}{\sum\limits_{l\in\mathbb{N}}\left(\frac{q}{2}\right)^{l-1}l^{-1}}\left(\sum_{k\in\mathbb{N}}\left(\frac{q}{2}\right)^{k-1}k^{-1}e_{kk}\right)E_q^{-1}.$$

Define $\iota_q : C\left(\overline{\mathbb{D}}_q\right) \longrightarrow \mathbb{C}$ by

$$\iota_q(a)=\operatorname{Tr}\left(A(q)aE_q\right)\text{ for }a\in C\left(\overline{\mathbb{D}}_q\right).$$

Naturally, ι_q is $\mathbb{C}$-linear. Since both $A(q)$ and E_q are diagonal, $\operatorname{Tr}\left(A(q)aE_q\right)\leq\|a\|\operatorname{Tr}\left(A(q)E_q\right)$ and $\operatorname{Tr}\left(A(q)a^*aE_q\right)=\operatorname{Tr}\left(\left(A(q)E_q\right)^{\frac{1}{2}}a^*a\left(A(q)E_q\right)^{\frac{1}{2}}\right)$. Thus, ι_q forms a well-defined positive linear functional on $C\left(\overline{\mathbb{D}}_q\right)$. By virtue of the choice of $A(q)$, $\iota_q(1)=2\pi$ for all $q\in[0,1)$, videlicet, $\iota_q(1)$ coincides with the surface area of a hemisphere of $\mathbb{S}^2$. Thence, as having done previously, we shall identify ι_q with the "noncommutative integration against the Riemannian volume form".

The left-hand side of the equation in Theorem 1.2.6 is called the total curvature of M for $n=1$. Thus, it suggests the notion that $\iota_q\left(R_{2112}E_q^{-2}\right)$ be deemed the total curvature of a quantum hemisphere of $C\left(\mathbb{S}^2_{qq}\right)$ or $C\left(\mathbb{CP}^1_{qq}\right)$. Thence, in accordance with the remark that succeeds Definition 1.3.9, the total curvature of $C\left(\mathbb{S}^2_{qq}\right)$ should correspondingly be

$$\iota_q\left(R_{2112}E_q^{-2}\right)-\iota_q\left(R_{2112}E_q^{-2}\right)=0$$

because its quantum disc components (hemispheres) are glued in such a way that they are "oppositely oriented", whereas that of $C\left(\mathbb{CP}^1_{qq}\right)$ should correspondingly be

$$\iota_q\left(R_{2112}E_q^{-2}\right)+\iota_q\left(R_{2112}E_q^{-2}\right)=2\iota_q\left(R_{2112}E_q^{-2}\right)$$

because its quantum disc components are glued in such a way that they are "consistently oriented". We shall demonstrate beneath that $\lim\limits_{q\to1^-}\iota_q\left(R_{2112}E_q^{-2}\right)=-2\pi$ and, thus, the total curvature of $C\left(\mathbb{CP}^1_{qq}\right)$ becomes -4π, the classical limit in Theorem 1.2.6, as $q\to1^-$.

Pursuant to Proposition 3.1.4,

$$g(\partial_{1;1},\partial_1)=\frac{1}{2}E_{q,1},\qquad g(\partial_{1;1},\partial_2)=-\frac{1}{2}E_{q,2},$$
$$g(\partial_{1;2},\partial_1)=\frac{1}{2}E_{q,2},\qquad g(\partial_{1;2},\partial_2)=\frac{1}{2}E_{q,1},$$

$$g(\partial_{2;1},\partial_2)=\frac{1}{2}E_{q,1}, \qquad g(\partial_{2;1},\partial_1)=\frac{1}{2}E_{q,2},$$
$$g(\partial_{2;2},\partial_2)=\frac{1}{2}E_{q,2}, \qquad g(\partial_{2;2},\partial_1)=-\frac{1}{2}E_{q,1}.$$

Thus, since $\{\partial_1,\partial_2\}$ is an orthogonal basis for $\Gamma\left(T^{\mathbb{C}}\overline{\mathbb{D}}_q\right)$,

$$\partial_{1;1}=\frac{1}{2}E_{q,1}E_q^{-1}\partial_1-\frac{1}{2}E_{q,2}E_q^{-1}\partial_2, \qquad \partial_{1;2}=\frac{1}{2}E_{q,2}E_q^{-1}\partial_1+\frac{1}{2}E_{q,1}E_q^{-1}\partial_2,$$
$$\partial_{2;1}=\frac{1}{2}E_{q,1}E_q^{-1}\partial_2+\frac{1}{2}E_{q,2}E_q^{-1}\partial_1, \qquad \partial_{2;2}=\frac{1}{2}E_{q,2}E_q^{-1}\partial_2-\frac{1}{2}E_{q,1}E_q^{-1}\partial_1.$$

Thence,

$$\begin{aligned}
&\iota_q\left(R_{2112}E_q^{-2}\right)\\
&=\iota_q\left(g\left((\partial_{1;2})_{;1}-(\partial_{1;1})_{;2}-\nabla_{[\partial_1,\partial_2]}\partial_1,\partial_2\right)E_q^{-2}\right)\\
&=\iota_q\left(g\left((\partial_{1;2})_{;1},\partial_2\right)E_q^{-2}\right)-\iota_q\left(g\left((\partial_{1;1})_{;2},\partial_2\right)E_q^{-2}\right)-\iota_q\left(g\left(\nabla_{[\partial_1,\partial_2]}\partial_1,\partial_2\right)E_q^{-2}\right)\\
&=\iota_q\left(g\left((\partial_{1;2})_{;1},\partial_2\right)E_q^{-2}\right)-\iota_q\left(g\left((\partial_{1;1})_{;2},\partial_2\right)E_q^{-2}\right)-\iota_q\left(g\left(\nabla_{\mathrm{ad}\left(i\frac{2}{1-q}Q\right)}\partial_1,\partial_2\right)E_q^{-2}\right)\\
&\quad\text{(by Lemma 3.1.3)}\\
&=\iota_q\left(g\left((\partial_{1;2})_{;1},\partial_2\right)E_q^{-2}\right)-\iota_q\left(g\left((\partial_{1;1})_{;2},\partial_2\right)E_q^{-2}\right)-\iota_q\left(g\left(i\frac{2}{1-q}Q\partial_1,\partial_2\right)E_q^{-2}\right)\\
&=\iota_q\left(g\left((\partial_{1;2})_{;1},\partial_2\right)E_q^{-2}\right)-\iota_q\left(g\left((\partial_{1;1})_{;2},\partial_2\right)E_q^{-2}\right)-\iota_q\left(0\cdot E_q^{-2}\right)\\
&=\iota_q\left(g\left(\left(\frac{1}{2}E_{q,2}E_q^{-1}\partial_1+\frac{1}{2}E_{q,1}E_q^{-1}\partial_2\right)_{;1},\partial_2\right)E_q^{-2}\right)-\\
&\quad\iota_q\left(g\left(\left(\frac{1}{2}E_{q,1}E_q^{-1}\partial_1-\frac{1}{2}E_{q,2}E_q^{-1}\partial_2\right)_{;2},\partial_2\right)E_q^{-2}\right)\\
&=\frac{1}{2}\iota_q\left(g\left(\left(E_{q,2}E_q^{-1}\partial_1\right)_{;1},\partial_2\right)E_q^{-2}\right)+\frac{1}{2}\iota_q\left(g\left(\left(E_{q,1}E_q^{-1}\partial_2\right)_{;1},\partial_2\right)E_q^{-2}\right)-\\
&\quad\frac{1}{2}\iota_q\left(g\left(\left(E_{q,1}E_q^{-1}\partial_1\right)_{;2},\partial_2\right)E_q^{-2}\right)+\frac{1}{2}\iota_q\left(g\left(\left(E_{q,2}E_q^{-1}\partial_2\right)_{;2},\partial_2\right)E_q^{-2}\right)\\
&=\frac{1}{2}\iota_q\left(\left(0-\frac{1}{2}E_{q,2}E_q^{-1}E_{q,2}\right)E_q^{-2}\right)+\frac{1}{2}\iota_q\left(\left(\left(E_{q,1}E_q^{-1}\right)_{,1}E_q+\frac{1}{2}E_{q,1}E_q^{-1}E_{q,1}\right)E_q^{-2}\right)-\\
&\quad\frac{1}{2}\iota_q\left(\left(0+\frac{1}{2}E_{q,1}E_q^{-1}E_{q,1}\right)E_q^{-2}\right)+\frac{1}{2}\iota_q\left(\left(\left(E_{q,2}E_q^{-1}\right)_{,2}E_q+\frac{1}{2}E_{q,2}E_q^{-1}E_{q,2}\right)E_q^{-2}\right)\\
&=\frac{1}{2}\iota_q\left(\left(E_{q,1}E_q^{-1}\right)_{,1}E_q^{-1}\right)+\frac{1}{2}\iota_q\left(\left(E_{q,2}E_q^{-1}\right)_{,2}E_q^{-1}\right)\\
&=\frac{1}{2}\mathrm{Tr}\left(A(q)\left(\left(E_{q,1}E_q^{-1}\right)_{,1}E_q^{-1}\right)E_q\right)+\frac{1}{2}\mathrm{Tr}\left(A(q)\left(\left(E_{q,2}E_q^{-1}\right)_{,2}E_q^{-1}\right)E_q\right)
\end{aligned}$$

$$= \frac{1}{2}\mathrm{Tr}\left(A(q)\left(E_{q,1}E_q^{-1}\right)_{,1}\right) + \frac{1}{2}\mathrm{Tr}\left(A(q)\left(E_{q,2}E_q^{-1}\right)_{,2}\right).$$

Via direct computation, we can obtain

$$\begin{aligned}
E_{q,1}E_q^{-1} &= -\frac{1}{i}\frac{2}{1-q}\left[E_q, y_q\right]E_q^{-1} \\
&= -\frac{1}{i}\frac{2}{1-q}\left[E_q, \frac{1}{2i}\left(z_q - z_q^*\right)\right]E_q^{-1} = \frac{1}{1-q}\left[E_q, z_q - z_q^*\right]E_q^{-1} \\
&= \frac{1}{1-q}\left(E_q\left(z_q - z_q^*\right) - \left(z_q - z_q^*\right)E_q\right)E_q^{-1} = \frac{1}{1-q}\left(\left(E_q z_q E_q^{-1} - z_q\right) - \left(E_q z_q^* E_q^{-1} - z_q^*\right)\right) \\
&= \frac{1}{1-q}\left(\left(\sum_{k\in\mathbb{N}}\frac{E_q^k}{E_q^{k-1}}e_{kk}\cdot z_q - z_q\right) - \left(\sum_{k\in\mathbb{N}}\frac{E_q^k}{E_q^{k+1}}e_{kk}\cdot z_q^* - z_q^*\right)\right) \\
&= \frac{1}{1-q}\left(\left(\sum_{k\in\mathbb{N}}\frac{E_q^k}{E_q^{k-1}}e_{kk} - 1\right)z_q - \left(\sum_{k\in\mathbb{N}}\frac{E_q^k}{E_q^{k+1}}e_{kk} - 1\right)z_q^*\right).
\end{aligned}$$

For lucidity and convenience, denote $\sum_{k\in\mathbb{N}}\frac{E_q^k}{E_q^{k-1}}e_{kk}$ and $\sum_{k\in\mathbb{N}}\frac{E_q^k}{E_q^{k+1}}e_{kk}$, respectively, by U and V. Naturally, U, V belong to $C\left(\overline{\mathbb{D}}_q\right)$. Then

$$\begin{aligned}
&\mathrm{Tr}\left(A(q)\left(E_{q,1}E_q^{-1}\right)_{,1}\right) \\
&= \mathrm{Tr}\left(A(q)\left(\frac{1}{1-q}\left((U-1)z_q - (V-1)z_q^*\right)\right)_{,1}\right) \\
&= \mathrm{Tr}\left(\frac{A(q)}{1-q}\left(\left(U_{,1}z_q + (U-1)Q\right) - \left(V_{,1}z_q^* + (V-1)Q\right)\right)\right) && \text{(by Lemma 3.1.2)} \\
&= \mathrm{Tr}\left(\frac{A(q)}{1-q}(U-V)Q + \frac{A(q)}{1-q}\left(U_{,1}z_q - V_{,1}z_q^*\right)\right) \\
&= \mathrm{Tr}\left(\frac{A(q)}{1-q}(U-V)Q + \frac{A(q)}{1-q}\left(-\frac{1}{i}\frac{2}{1-q}\left[U, y_q\right]z_q - \left(-\frac{1}{i}\frac{2}{1-q}\left[V, y_q\right]\right)z_q^*\right)\right) \\
&= \mathrm{Tr}\left(\frac{A(q)}{1-q}(U-V)Q + i\frac{2A(q)}{(1-q)^2}\left(\left[U, y_q\right]z_q - \left[V, y_q\right]z_q^*\right)\right) \\
&= \mathrm{Tr}\left(\frac{A(q)}{1-q}(U-V)Q + i\frac{2A(q)}{(1-q)^2}\left(\left[U, \frac{1}{2i}\left(z_q - z_q^*\right)\right]z_q - \left[V, \frac{1}{2i}\left(z_q - z_q^*\right)\right]z_q^*\right)\right) \\
&= \mathrm{Tr}\left(\frac{A(q)}{1-q}(U-V)Q + \frac{A(q)}{(1-q)^2}\left(\left[U, z_q - z_q^*\right]z_q - \left[V, z_q - z_q^*\right]z_q^*\right)\right) \\
&= \mathrm{Tr}\Bigg(\frac{A(q)}{1-q}(U-V)Q + \frac{A(q)}{(1-q)^2}\Bigg(\sum_{k\in\mathbb{N}}\sqrt{1-q^k}\left(\frac{E_q^{k+1}}{E_q^k} - \frac{E_q^k}{E_q^{k-1}}\right)\left(e_{k+1,k} + e_{k,k+1}\right)\cdot z_q -
\end{aligned}$$

$$\sum_{k\in\mathbb{N}}\sqrt{1-q^k}\left(\frac{E_q^{k+1}}{E_q^{k+2}}-\frac{E_q^k}{E_q^{k+1}}\right)(e_{k+1,k}+e_{k,k+1})\cdot z_q^*\Bigg)\Bigg)$$

$$=\mathrm{Tr}\Bigg(\frac{A(q)}{1-q}(U-V)Q+\frac{A(q)}{(1-q)^2}\Bigg(\sum_{k\in\mathbb{N}}\left(1-q^k\right)\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k-1}}\right)e_{kk}-$$

$$\sum_{k\in\mathbb{N}}\left(1-q^k\right)\left(\frac{E_q^{k+1}}{E_q^{k+2}}-\frac{E_q^k}{E_q^{k+1}}\right)e_{k+1,k+1}\Bigg)\Bigg)$$

$$=\mathrm{Tr}\Bigg(\frac{A(q)}{1-q}(U-V)Q+\frac{A(q)}{(1-q)^2}\Bigg((1-q)\left(\frac{E_q^2}{E_q^1}-\frac{E_q^1}{E_q^0}\right)e_{11}+$$

$$\sum_{k\geq 2}\left(\left(1-q^k\right)\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k-1}}\right)-\left(1-q^{k-1}\right)\left(\frac{E_q^k}{E_q^{k+1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right)e_{kk}\Bigg)\Bigg)$$

$$=\mathrm{Tr}\Bigg(\frac{A(q)}{1-q}\left(\sum_{k\in\mathbb{N}}q^{k-1}\left(\frac{E_q^k}{E_q^{k-1}}-\frac{E_q^k}{E_q^{k+1}}\right)e_{kk}\right)+\frac{A(q)}{(1-q)^2}\Bigg((1-q)(\frac{E_q^2}{E_q^1}-\frac{E_q^1}{E_q^0})e_{11}+$$

$$\sum_{k\geq 2}\left(\left(1-q^k\right)\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k-1}}\right)-\left(1-q^{k-1}\right)\left(\frac{E_q^k}{E_q^{k+1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right)e_{kk}\Bigg)\Bigg)$$

$$=\mathrm{Tr}\Bigg(\frac{A(q)}{(1-q)^2}\Bigg((1-q)\left(\frac{E_q^2}{E_q^1}-\frac{E_q^1}{E_q^2}\right)e_{11}+$$

$$\sum_{k\geq 2}\left(\left(1-q^k\right)\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)-\left(1-q^{k-1}\right)\left(\frac{E_q^k}{E_q^{k-1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right)e_{kk}\Bigg)\Bigg)$$

$$=\mathrm{Tr}\Bigg(\frac{A(q)}{1-q}\left(\frac{E_q^2}{E_q^1}-\frac{E_q^1}{E_q^2}\right)e_{11}+$$

$$\sum_{k\geq 2}\left(\sum_{l=0}^{k-1}q^l\cdot\frac{A(q)}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)-\sum_{l=0}^{k-2}q^l\cdot\frac{A(q)}{1-q}\left(\frac{E_q^k}{E_q^{k-1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right)e_{kk}\Bigg).$$

Similarly, $\mathrm{Tr}\left(A(q)\left(E_{q,2}E_q^{-1}\right)_{,2}\right)=$

$$\mathrm{Tr}\Bigg(\frac{A(q)}{1-q}\left(\frac{E_q^2}{E_q^1}-\frac{E_q^1}{E_q^2}\right)e_{11}+$$

$$\sum_{k\geq 2}\left(\sum_{l=0}^{k-1}q^l\cdot\frac{A(q)}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)-\sum_{l=0}^{k-2}q^l\cdot\frac{A(q)}{1-q}\left(\frac{E_q^k}{E_q^{k-1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right)e_{kk}\Bigg).$$

Thence, $\mathrm{Tr}\left(A(q)\left(E_{q,2}E_q^{-1}\right)_{,2}\right)=\mathrm{Tr}\left(A(q)\left(E_{q,1}E_q^{-1}\right)_{,1}\right)$ and, thus, $\iota_q\left(R_{2112}E_q^{-2}\right)=\mathrm{Tr}\left(A(q)\left(E_{q,1}E_q^{-1}\right)_{,1}\right)$.

Via direct computation, we can obtain

$$
\begin{aligned}
&\frac{1}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)\\
&=\frac{1}{1-q}\left(E_q^{k+1}-E_q^k\right)\frac{E_q^{k+1}+E_q^k}{E_q^kE_q^{k+1}}\\
&=\frac{1}{1-q}\left(\frac{16}{\left(4-q^k-q^{k+1}\right)^2}-\frac{16}{\left(4-q^{k-1}-q^k\right)^2}\right)\frac{E_q^{k+1}+E_q^k}{E_q^kE_q^{k+1}}\\
&=\frac{1}{1-q}\left(\frac{4}{4-q^k-q^{k+1}}-\frac{4}{4-q^{k-1}-q^k}\right)\left(\frac{4}{4-q^k-q^{k+1}}+\frac{4}{4-q^{k-1}-q^k}\right)\frac{E_q^{k+1}+E_q^k}{E_q^kE_q^{k+1}}\\
&=-\frac{4q^{k-1}(q+1)}{\left(4-q^k-q^{k+1}\right)\left(4-q^{k-1}-q^k\right)}\left(\frac{4}{4-q^k-q^{k+1}}+\frac{4}{4-q^{k-1}-q^k}\right)\frac{E_q^{k+1}+E_q^k}{E_q^kE_q^{k+1}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|\sum_{l=0}^{k-1}q^l\cdot\frac{f(q)_k}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)-\sum_{l=0}^{k-2}q^l\cdot\frac{f(q)_k}{1-q}\left(\frac{E_q^k}{E_q^{k-1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right|\\
&\le\left|\sum_{l=0}^{k-1}q^l\cdot\frac{f(q)_k}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)-0\right|\\
&\le\sum_{l=0}^{k-1}q^l\cdot f(q)_k\left|\frac{1}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)\right|\\
&\le\sum_{l=0}^{k-1}1^l\cdot f(q)_k\cdot\frac{4\cdot 1^{k-1}\cdot(1+1)}{\left(4-1^k-1^{k+1}\right)\left(4-1^{k-1}-1^k\right)}\left(\frac{4}{4-1^k-1^{k+1}}+\frac{4}{4-1^{k-1}-1^k}\right)\frac{4+4}{1\cdot 1}\\
&=64kf(q)_k\\
&\le 64g_k\text{ for all }q\in(1-\varepsilon,1).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\lim_{q\to 1^-}\frac{1}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)\\
&=-\frac{4\cdot 1^{k-1}\cdot(1+1)}{\left(4-1^k-1^{k+1}\right)\left(4-1^{k-1}-1^k\right)}\left(\frac{4}{4-1^k-1^{k+1}}+\frac{4}{4-1^{k-1}-1^k}\right)\frac{4+4}{4\cdot 4}\\
&=-4.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{q\to 1^-} \iota_q\left(R_{2112}E_q^{-2}\right) \\
&= \lim_{q\to 1^-} \operatorname{Tr}\Big(A(q)\left(E_{q,1}E_q^{-1}\right)_{,1}\Big) \\
&= \lim_{q\to 1^-} \operatorname{Tr}\Bigg(\frac{A(q)}{1-q}\left(\frac{E_q^2}{E_q^1}-\frac{E_q^1}{E_q^2}\right)e_{11}+ \\
&\qquad \sum_{k\geq 2}\left(\sum_{l=0}^{k-1} q^l\cdot\frac{A(q)}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)-\sum_{l=0}^{k-2} q^l\cdot\frac{A(q)}{1-q}\left(\frac{E_q^k}{E_q^{k-1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right)e_{kk}\Bigg) \\
&= \lim_{q\to 1^-}\Bigg(\frac{f(q)_1}{1-q}\left(\frac{E_q^2}{E_q^1}-\frac{E_q^1}{E_q^2}\right)+ \\
&\qquad \sum_{k\geq 2}\left(\sum_{l=0}^{k-1} q^l\cdot\frac{f(q)_k}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)-\sum_{l=0}^{k-2} q^l\cdot\frac{f(q)_k}{1-q}\left(\frac{E_q^k}{E_q^{k-1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right)\Bigg) \\
&= \lim_{q\to 1^-}\frac{f(q)_1}{1-q}\left(\frac{E_q^2}{E_q^1}-\frac{E_q^1}{E_q^2}\right)+ \\
&\quad \sum_{k\geq 2}\lim_{q\to 1^-}\left(\sum_{l=0}^{k-1} q^l\cdot\frac{f(q)_k}{1-q}\left(\frac{E_q^{k+1}}{E_q^k}-\frac{E_q^k}{E_q^{k+1}}\right)-\sum_{l=0}^{k-2} q^l\cdot\frac{f(q)_k}{1-q}\left(\frac{E_q^k}{E_q^{k-1}}-\frac{E_q^{k-1}}{E_q^k}\right)\right) \\
&\quad \text{(by Lebesgue's Dominated Convergence Theorem (Theorem 2.24 in [7]))} \\
&= \lim_{q\to 1^-} f(q)_1\cdot(-4)+\sum_{k\geq 2}\left(k\cdot\lim_{q\to 1^-} f(q)_k\cdot(-4)-(k-1)\cdot\lim_{q\to 1^-} f(q)_k\cdot(-4)\right) \\
&= -\lim_{q\to 1^-} f(q)_1\cdot 4-\sum_{k\geq 2}\left(\lim_{q\to 1^-} f(q)_k\cdot 4\right) \\
&= -\lim_{q\to 1^-} f(q)_1\cdot\lim_{q\to 1^-} E_q^1-\sum_{k\geq 2}\left(\lim_{q\to 1^-} f(q)_k\cdot\lim_{q\to 1^-} E_q^k\right) \\
&= -\lim_{q\to 1^-} f(q)_1 E_q^1-\sum_{k\geq 2}\lim_{q\to 1^-} f(q)_k E_q^k \\
&= -\lim_{q\to 1^-}\left(f(q)_1E_q^1+\sum_{k\geq 2} f(q)_k E_q^k\right) \qquad \text{(by Lebesgue's Dominated Convergence Theorem)} \\
&= -\lim_{q\to 1^-}\operatorname{Tr}\left(A(q)E_q\right) \\
&= -2\pi.
\end{aligned}
$$

In particular, the total curvature of $C\left(\mathbb{CP}^1_{qq}\right)$ becomes -4π as $q\to 1^-$.

We recapitulate beneath what we have attained.

Proposition 3.2.1. Let g be a Riemannian metric on $C\left(\overline{\mathbb{D}}_q\right)$. Suppose further that

$$\left[g_{kl}\right] = \mathrm{Diag}(E_q, E_q).$$

Then

$$\lim_{q \to 1^-} \iota_q\left(R_{2112}E_q^{-2}\right) = -2\pi.$$

In particular, the total curvature of $C\left(\mathbb{CP}^1_{qq}\right)$ becomes -4π as $q \to 1^-$.

3.3 Prospects

We have hitherto only attained an asymptotic behaviour of the total curvature for quantum complex projective lines in lieu of a noncommutative version of the Gauß-Bonnet Theorem as in the case of noncommutative tori demonstrated in Chapter 2. We thus surmise that some other theories might need to be introduced in to achieve stronger results. One such possibility abides in A. J.-L. Sheu's classification of the projective modules over $C\left(\mathbb{CP}^1_{qq}\right)$ ([20]), which suggests the notion that two such modules be deemed the noncommutative tangent bundle of $C\left(\mathbb{CP}^1_{qq}\right)$ because they are the noncommutative analogues of the complex line bundles over $\mathbb{CP}^1$ that embody the tangent bundle of $\mathbb{CP}^1$ as a smooth real manifold. We have not yet discern how to incorporate this conception into our work. Notwithstanding, it advises a new method that might prove to be assistance to future research.

Appendix A

4-dimensional Noncommutative Tori

Proposition. For $n = 4$, let g be a Riemannian metric on $C^\infty(\mathbb{T}_\Theta)$. Suppose further that

$$\left[g_{kl}\right] = \mathrm{Diag}(a,a,a,a)$$

for some $a \in C^\infty(\mathbb{T}_\Theta)_{\geq}$. Define $\widehat{\mathrm{Pf}}_\Theta\left(R_{\alpha\beta\gamma\delta}\right)$ to be

$$\begin{gathered}\left(R_{3434}R_{1212} - R_{1312}R^*_{2434} + R_{1412}R^*_{2334} + R_{2312}R^*_{1434} - R_{2412}R^*_{1334} + R_{3412}R_{1234}\right) + \\ \left(-R_{1213}R^*_{3424} + R_{2424}R_{1313} - R_{1413}R_{2324} - R^*_{2313}R^*_{1424} + R_{2413}R_{1324} - R_{3413}R^*_{1224}\right) + \\ \left(R_{1214}R^*_{3423} - R_{1314}R_{2423} + R_{2323}R_{1414} + R_{2314}R_{1423} - R^*_{2414}R^*_{1323} + R^*_{3414}R^*_{1223}\right).\end{gathered}$$

If $a_{,k}a = aa_{,k}$ for all $1 \leq k \leq 4$, and, moreover,

(a) $a_{,1}{}^2a^{-1}a_{,kk} = a_{,kk}a_{,1}{}^2a^{-1}$ for all $1 \leq k \leq 4$,

(b) $a_{,k}{}^2a^{-1}a_{,11} = a_{,k}{}^2a_{,11}a^{-1}$ for all $2 \leq k \leq 4$,

(c) $a_{,k}a_{,1}a^{-1}a_{,1k} + \frac{1}{2}a_{,1}a_{,k}a^{-1}a_{,1k} - a_{,k1}a_{,1}a_{,k}a^{-1} =$
$\frac{5}{4}a_{,k}a_{,1k}a_{,1}a^{-1} + \frac{5}{4}a_{,k}a_{,1}a_{,1k}a^{-1} - \frac{5}{4}a_{,k1}a_{,k}a_{,1}a^{-1} - \frac{3}{4}a_{,k}a_{,k1}a_{,1}a^{-1}$ for all $2 \leq k \leq 4$,

(d) $\frac{1}{2}a_{,1}a_{,k}a_{,1}a_{,k} + \frac{1}{2}a_{,k}a_{,1}a_{,k}a_{,1} + \frac{1}{4}a_{,1}a_{,k}{}^2a_{,1} = \frac{11}{4}a_{,k}a_{,1}{}^2a_{,k} - \frac{3}{2}a_{,k}{}^2a_{,1}{}^2$ for all $2 \leq k \leq 4$,

(e) $a_{,k}{}^2a^{-1}a_{,ll} = a_{,ll}a_{,k}{}^2a^{-1}$ for all $2 \leq k,l \leq 4$ with $l \neq k$,

(f) $a_{,l}a_{,k}a^{-1}a_{,kl} + \frac{1}{2}a_{,k}a_{,l}a^{-1}a_{,kl} - a_{,lk}a_{,k}a_{,l}a^{-1} + \frac{1}{2}a_{,lk}a_{,l}a_{,k}a^{-1} + a_{,k}a_{,l}a^{-1}a_{,lk} + \frac{1}{2}a_{,l}a_{,k}a^{-1}a_{,lk} -$
$a_{,kl}a_{,l}a_{,k}a^{-1} + \frac{1}{2}a_{,kl}a_{,k}a_{,l}a^{-1} =$
$\frac{1}{2}\left(a_{,l}a_{,kl}a_{,k}a^{-1} + a_{,l}a_{,k}a_{,kl}a^{-1} + a_{,k}a_{,lk}a_{,j}a^{-1} + a_{,k}a_{,l}a_{,lk}a^{-1}\right)$ for all $2 \leq k < l \leq 4$,

(g) $\frac{3}{4}a_{,l}{}^2a_{,k}{}^2 + \frac{3}{4}a_{,k}{}^2a_{,l}{}^2 - a_{,k}a_{,l}a_{,k}a_{,l} - a_{,l}a_{,k}a_{,l}a_{,k} = -\frac{1}{4}\left(a_{,l}a_{,k}{}^2a_{,l} + a_{,k}a_{,l}{}^2a_{,k}\right)$ for all $2 \leq k <$
$l \leq 4$,

(h) $3a_{,1}{}^2a^{-1}a_{,11} = a_{,11}a_{,1}{}^2a^{-1} + a_{,1}a_{,11}a_{,1}a^{-1} + a_{,1}{}^2a_{,11}a^{-1}$, and

(i) $a_{,k}{}^2 a^{-1} a_{,kk} + a_{,kk} a_{,k}{}^2 a^{-1} = a_{,k} a_{,kk} a_{,k} a^{-1} + a_{,k}{}^2 a_{,kk} a^{-1}$ for all $2 \le k \le 4$,

then

$$\tau\left(\widehat{\mathrm{Pf}}_{\Theta}\left(R_{\alpha\beta\gamma\delta}\right) a^{-2}\right) = 0.$$

Proof. Pursuant to the metric hypothesis,

$$\partial_{i;i} = \frac{1}{2}\left(a_{,i}a^{-1}\partial_i - \sum_{\substack{1\le k\le 4\\ k\ne i}} a_{,k}a^{-1}\partial_k\right), \qquad 1 \le i \le 4,$$

$$\partial_{i;j} = \frac{1}{2}\left(a_{,j}a^{-1}\partial_i + a_{,i}a^{-1}\partial_j\right), \qquad 1 \le i, j \le 4 \text{ with } j \ne i.$$

Thus,

$$\begin{aligned}
R_{ijkl} &= g\left(\left(\partial_{j;l}\right)_{;k}, \partial_i\right) - g\left(\left(\partial_{j;k}\right)_{;l}, \partial_i\right)\\
&= \frac{1}{2}(0+0+0+0) - \frac{1}{2}(0+0+0+0) = 0, \qquad 1 \le i,j,k,l \le 4 \text{ with } i,j,k,l \text{ distinct},\\
R_{ijij} &= g\left(\left(\partial_{j;j}\right)_{;i}, \partial_i\right) - g\left(\left(\partial_{j;i}\right)_{;j}, \partial_i\right)\\
&= \frac{1}{2}\left(0 + \frac{1}{2}a_{,j}a^{-1}a_{,j} - \left(a_{,i}a^{-1}\right)_{,i} a - \frac{1}{2}a_{,i}a^{-1}a_{,i} - \sum_{\substack{1\le k\le 4\\ k\ne i,j}}\left(0 + \frac{1}{2}a_{,k}a^{-1}a_{,k}\right)\right) -\\
&\quad \frac{1}{2}\left(0 - \frac{1}{2}a_{,i}a^{-1}a_{,i} + \left(a_{,j}a^{-1}\right)_{,j} a + \frac{1}{2}a_{,j}a^{-1}a_{j}\right)\\
&= -\frac{1}{2}\left(\left(a_{,ii} - a_{,i}{}^2a^{-1}\right) + \left(a_{,jj} - a_{,j}{}^2a^{-1}\right) + \sum_{\substack{1\le k\le 4\\ k\ne i,j}} \frac{1}{2}a_{,k}{}^2a^{-1}\right), \qquad 1 \le i,j \le 4 \text{ with } j \ne i,\\
R_{ijil} &= g\left(\left(\partial_{j;l}\right)_{;i}, \partial_i\right) - g\left(\left(\partial_{j;i}\right)_{;l}, \partial_i\right)\\
&= \frac{1}{2}\left(0 + \frac{1}{2}a_{,l}a^{-1}a_{,j} + 0 + \frac{1}{2}a_{,j}a^{-1}a_{,l}\right) - \frac{1}{2}\left(0 + 0 + \left(a_{,j}a^{-1}\right)_{,l} a + \frac{1}{2}a_{,j}a^{-1}a_{,l}\right)\\
&= -\frac{1}{2}\left(\left(a_{,jl} - a_{,j}a_{,l}a^{-1}\right) - \frac{1}{2}a_{,l}a_{,j}a^{-1}\right), \qquad 1 \le i,j,l \le 4 \text{ with } i,j,l \text{ distinct},\\
R_{ijjl} &= g\left(\left(\partial_{j;l}\right)_{;j}, \partial_i\right) - g\left(\left(\partial_{j;j}\right)_{;l}, \partial_i\right)\\
&= \frac{1}{2}\left(0 - \frac{1}{2}a_{,l}a^{-1}a_{,i} + 0 + 0\right) -\\
&\quad \frac{1}{2}\left(0 + 0 - \left(a_{,i}a^{-1}\right)_{,l} a - \frac{1}{2}a_{,i}a^{-1}a_{,l} - 0 + \frac{1}{2}a_{,l}a^{-1}a_{,i} - 0 - 0\right)\\
&= \frac{1}{2}\left(\left(a_{,il} - \frac{1}{2}a_{,i}a_{,l}a^{-1}\right) - a_{,l}a_{,i}a^{-1}\right), \qquad 1 \le i,j,l \le 4 \text{ with } i,j,l \text{ distinct},
\end{aligned}$$

$$R_{ijkj} = -R_{ijjk} \qquad \text{(by Proposition 2.1.3)}$$

$$= -\frac{1}{2}\left(\left(a_{,ik} - \frac{1}{2}a_{,i}a_{,k}a^{-1}\right) - a_{,k}a_{,i}a^{-1}\right), \qquad 1 \le i,j,k \le 4 \text{ with } i,j,k \text{ distinct,}$$

$$R_{ijki} = -R_{ijik} \qquad \text{(by Proposition 2.1.3)}$$

$$= \frac{1}{2}\left(\left(a_{,jk} - a_{,j}a_{,k}a^{-1}\right) - \frac{1}{2}a_{,k}a_{,j}a^{-1}\right), \qquad 1 \le i,j,k \le 4 \text{ with } i,j,k \text{ distinct.}$$

Thence,

$$\begin{aligned}
&\widehat{\mathrm{Pf}}_{\Theta}\left(R_{\alpha\beta\gamma\delta}\right) \\
&= \frac{1}{4}\Bigg(\left(a_{,33} - a_{,3}{}^2a^{-1} + a_{,44} - a_{,4}{}^2a^{-1} + \frac{1}{2}a_{,1}{}^2a^{-1} + \frac{1}{2}a_{,2}{}^2a^{-1}\right) \times \\
&\qquad \left(a_{,11} - a_{,1}{}^2a^{-1} + a_{,22} - a_{,2}{}^2a^{-1} + \frac{1}{2}a_{,3}{}^2a^{-1} + \frac{1}{2}a_{,4}{}^2a^{-1}\right) - \\
&\qquad \left(a_{,32} - a_{,3}a_{,2}a^{-1} - \frac{1}{2}a_{,2}a_{,3}a^{-1})(a_{,23} - a_{,2}a_{,3}a^{-1} - \frac{1}{2}a_{,3}a_{,2}a^{-1}\right) - \\
&\qquad \left(a_{,42} - a_{,4}a_{,2}a^{-1} - \frac{1}{2}a_{,2}a_{,4}a^{-1})(a_{,24} - a_{,2}a_{,4}a^{-1} - \frac{1}{2}a_{,4}a_{,2}a^{-1}\right) - \\
&\qquad \left(a_{,31} - a_{,3}a_{,1}a^{-1} - \frac{1}{2}a_{,1}a_{,3}a^{-1})(a_{,13} - a_{,1}a_{,3}a^{-1} - \frac{1}{2}a_{,3}a_{,1}a^{-1}\right) - \\
&\qquad \left(a_{,41} - a_{,4}a_{,1}a^{-1} - \frac{1}{2}a_{,1}a_{,4}a^{-1})(a_{,14} - a_{,1}a_{,4}a^{-1} - \frac{1}{2}a_{,4}a_{,1}a^{-1}\right) + 0\Bigg) + \\
&\quad \frac{1}{4}\Bigg(-\left(a_{,23} - a_{,2}a_{,3}a^{-1} - \frac{1}{2}a_{,3}a_{,2}a^{-1})(a_{,32} - a_{,3}a_{,2}a^{-1} - \frac{1}{2}a_{,2}a_{,3}a^{-1}\right) + \\
&\qquad \left(a_{,22} - a_{,2}{}^2a^{-1} + a_{,44} - a_{,4}{}^2a^{-1} + \frac{1}{2}a_{,1}{}^2a^{-1} + \frac{1}{2}a_{,3}{}^2a^{-1}\right) \times \\
&\qquad \left(a_{,11} - a_{,1}{}^2a^{-1} + a_{,33} - a_{,3}{}^2a^{-1} + \frac{1}{2}a_{,2}{}^2a^{-1} + \frac{1}{2}a_{,4}{}^2a^{-1}\right) - \\
&\qquad \left(a_{,43} - a_{,4}a_{,3}a^{-1} - \frac{1}{2}a_{,3}a_{,4}a^{-1})(a_{,34} - a_{,3}a_{,4}a^{-1} - \frac{1}{2}a_{,4}a_{,3}a^{-1}\right) - \\
&\qquad \left(a_{,21} - a_{,2}a_{,1}a^{-1} - \frac{1}{2}a_{,1}a_{,2}a^{-1})(a_{,12} - a_{,1}a_{,2}a^{-1} - \frac{1}{2}a_{,2}a_{,1}a^{-1}\right) + 0 - \\
&\qquad \left(a_{,41} - a_{,4}a_{,1}a^{-1} - \frac{1}{2}a_{,1}a_{,4}a^{-1})(a_{,14} - a_{,1}a_{,4}a^{-1} - \frac{1}{2}a_{,4}a_{,1}a^{-1}\right)\Bigg) + \\
&\quad \frac{1}{4}\Bigg(-\left(a_{,24} - a_{,2}a_{,4}a^{-1} - \frac{1}{2}a_{,4}a_{,2}a^{-1})(a_{,42} - a_{,4}a_{,2}a^{-1} - \frac{1}{2}a_{,2}a_{,4}a^{-1}\right) - \\
&\qquad \left(a_{,34} - a_{,3}a_{,4}a^{-1} - \frac{1}{2}a_{,4}a_{,3}a^{-1})(a_{,43} - a_{,4}a_{,3}a^{-1} - \frac{1}{2}a_{,3}a_{,4}a^{-1}\right) + \\
&\qquad \left(a_{,22} - a_{,2}{}^2a^{-1} + a_{,33} - a_{,3}{}^2a^{-1} + \frac{1}{2}a_{,1}{}^2a^{-1} + \frac{1}{2}a_{,4}{}^2a^{-1}\right) \times
\end{aligned}$$

$$
\begin{aligned}
&\left(a_{,11}-a_{,1}{}^2a^{-1}+a_{,44}-a_{,4}{}^2a^{-1}+\frac{1}{2}a_{,2}{}^2a^{-1}+\frac{1}{2}a_{,3}{}^2a^{-1}\right)+0-\\
&\left(a_{,21}-a_{,2}a_{,1}a^{-1}-\frac{1}{2}a_{,1}a_{,2}a^{-1})(a_{,12}-a_{,1}a_{,2}a^{-1}-\frac{1}{2}a_{,2}a_{,1}a^{-1}\right)-\\
&\left(a_{,31}-a_{,3}a_{,1}a^{-1}-\frac{1}{2}a_{,1}a_{,3}a^{-1})(a_{,13}-a_{,1}a_{,3}a^{-1}-\frac{1}{2}a_{,3}a_{,1}a^{-1}\right)\Bigg)\\
=&\frac{1}{2}\left(a_{,221}a_{,1}+a_{,22}a_{,11}-2a_{,22}a_{,1}{}^2a^{-1}\right)+\frac{1}{2}\left(a_{,331}a_{,1}+a_{,33}a_{,11}-2a_{,33}a_{,1}{}^2a^{-1}\right)+\\
&\frac{1}{2}\left(a_{,441}a_{,1}+a_{,44}a_{,11}-2a_{,44}a_{,1}{}^2a^{-1}\right)-\frac{1}{2}\left(a_{,212}a_{,1}+a_{,21}a_{,12}-2a_{,21}a_{,1}a_{,2}a^{-1}\right)-\\
&\frac{1}{2}\left(a_{,313}a_{,1}+a_{,31}a_{,13}-2a_{,31}a_{,1}a_{,3}a^{-1}\right)-\frac{1}{2}\left(a_{,414}a_{,1}+a_{,41}a_{,14}-2a_{,41}a_{,1}a_{,4}a^{-1}\right)+\\
&\frac{5}{8}\left(a_{,22}a_{,1}{}^2a^{-1}+a_{,2}a_{,12}a_{,1}a^{-1}+a_{,2}a_{,1}a_{,12}a^{-1}-3a_{,2}a_{,1}{}^2a_{,2}a^{-2}\right)+\\
&\frac{5}{8}\left(a_{,33}a_{,1}{}^2a^{-1}+a_{,3}a_{,13}a_{,1}a^{-1}+a_{,3}a_{,1}a_{,13}a^{-1}-3a_{,3}a_{,1}{}^2a_{,3}a^{-2}\right)+\\
&\frac{5}{8}\left(a_{,44}a_{,1}{}^2a^{-1}+a_{,4}a_{,14}a_{,1}a^{-1}+a_{,4}a_{,1}a_{,14}a^{-1}-3a_{,4}a_{,1}{}^2a_{,4}a^{-2}\right)-\\
&\frac{3}{8}\left(a_{,21}a_{,2}a_{,1}a^{-1}+a_{,2}a_{,21}a_{,1}a^{-1}+a_{,2}{}^2a_{,11}a^{-1}-3a_{,2}{}^2a_{,1}{}^2a^{-2}\right)-\\
&\frac{3}{8}\left(a_{,31}a_{,3}a_{,1}a^{-1}+a_{,3}a_{,31}a_{,1}a^{-1}+a_{,3}{}^2a_{,11}a^{-1}-3a_{,3}{}^2a_{,1}{}^2a^{-2}\right)-\\
&\frac{3}{8}\left(a_{,41}a_{,4}a_{,1}a^{-1}+a_{,4}a_{,41}a_{,1}a^{-1}+a_{,4}{}^2a_{,11}a^{-1}-3a_{,4}{}^2a_{,1}{}^2a^{-2}\right)+\\
&\frac{1}{4}\left(a_{,332}a_{,2}+a_{,33}a_{,22}-2a_{,33}a_{,2}{}^2a^{-1}\right)+\frac{1}{4}\left(a_{,442}a_{,2}+a_{,44}a_{,22}-2a_{,44}a_{,2}{}^2a^{-1}\right)+\\
&\frac{1}{4}\left(a_{,443}a_{,2}+a_{,44}a_{,33}-2a_{,44}a_{,3}{}^2a^{-1}\right)+\frac{1}{4}\left(a_{,223}a_{,3}+a_{,22}a_{,33}-2a_{,22}a_{,3}{}^2a^{-1}\right)+\\
&\frac{1}{4}\left(a_{,224}a_{,4}+a_{,22}a_{,44}-2a_{,22}a_{,4}{}^2a^{-1}\right)+\frac{1}{4}\left(a_{,334}a_{,4}+a_{,33}a_{,44}-2a_{,33}a_{,4}{}^2a^{-1}\right)-\\
&\frac{1}{4}\left(a_{,323}a_{,2}+a_{,32}a_{,23}-2a_{,32}a_{,2}a_{,3}a^{-1}\right)-\frac{1}{4}\left(a_{,424}a_{,2}+a_{,42}a_{,24}-2a_{,42}a_{,2}a_{,4}a^{-1}\right)-\\
&\frac{1}{4}\left(a_{,434}a_{,3}+a_{,43}a_{,34}-2a_{,43}a_{,3}a_{,4}a^{-1}\right)-\frac{1}{4}\left(a_{,232}a_{,3}+a_{,23}a_{,32}-2a_{,23}a_{,3}a_{,2}a^{-1}\right)-\\
&\frac{1}{4}\left(a_{,242}a_{,4}+a_{,24}a_{,42}-2a_{,24}a_{,4}a_{,2}a^{-1}\right)-\frac{1}{4}\left(a_{,343}a_{,4}+a_{,34}a_{,43}-2a_{,34}a_{,4}a_{,3}a^{-1}\right)+\\
&\frac{1}{8}\left(a_{,33}a_{,2}{}^2a^{-1}+a_{,3}a_{,23}a_{,2}a^{-1}+a_{,3}a_{,2}a_{,23}a^{-1}-3a_{,3}a_{,2}{}^2a_{,3}a^{-2}\right)+\\
&\frac{1}{8}\left(a_{,44}a_{,2}{}^2a^{-1}+a_{,4}a_{,24}a_{,2}a^{-1}+a_{,4}a_{,2}a_{,24}a^{-1}-3a_{,4}a_{,2}{}^2a_{,4}a^{-2}\right)+\\
&\frac{1}{8}\left(a_{,44}a_{,3}{}^2a^{-1}+a_{,4}a_{,34}a_{,3}a^{-1}+a_{,4}a_{,3}a_{,34}a^{-1}-3a_{,4}a_{,3}{}^2a_{,4}a^{-2}\right)+\\
&\frac{1}{8}\left(a_{,22}a_{,3}{}^2a^{-1}+a_{,2}a_{,32}a_{,3}a^{-1}+a_{,2}a_{,3}a_{,32}a^{-1}-3a_{,2}a_{,3}{}^2a_{,2}a^{-2}\right)+
\end{aligned}
$$

$$\frac{1}{8}\left(a_{,22}a_{,4}{}^{2}a^{-1}+a_{,2}a_{,42}a_{,3}a^{-1}+a_{,2}a_{,4}a_{,42}a^{-1}-3a_{,2}a_{,4}{}^{2}a_{,2}a^{-2}\right)+$$
$$\frac{1}{8}\left(a_{,33}a_{,4}{}^{2}a^{-1}+a_{,3}a_{,43}a_{,4}a^{-1}+a_{,3}a_{,4}a_{,43}a^{-1}-3a_{,3}a_{,4}{}^{2}a_{,3}a^{-2}\right)+$$
$$\frac{1}{8}\left(a_{,11}a_{,1}{}^{2}a^{-1}+a_{,1}a_{,11}a_{,1}a^{-1}+a_{,1}{}^{2}a_{,11}a^{-1}-3a_{,1}{}^{4}a^{-2}\right)+$$
$$\frac{1}{8}\left(a_{,22}a_{,2}{}^{2}a^{-1}+a_{,2}a_{,22}a_{,2}a^{-1}+a_{,2}{}^{2}a_{,22}a^{-1}-3a_{,2}{}^{4}a^{-2}\right)+$$
$$\frac{1}{8}\left(a_{,33}a_{,3}{}^{2}a^{-1}+a_{,3}a_{,33}a_{,3}a^{-1}+a_{,3}{}^{2}a_{,33}a^{-1}-3a_{,3}{}^{4}a^{-2}\right)+$$
$$\frac{1}{8}\left(a_{,44}a_{,4}{}^{2}a^{-1}+a_{,4}a_{,44}a_{,4}a^{-1}+a_{,4}{}^{2}a_{,44}a^{-1}-3a_{,4}{}^{4}a^{-2}\right).$$

Therefore, since τ is invariant under the group action generated by $\{\partial_k\}_{k=1}^{4}$,

$$\begin{aligned}
&\tau\left(\widehat{\mathrm{Pf}}_{\Theta}\left(R_{\alpha\beta\gamma\delta}\right)a^{-2}\right)\\
&=-\frac{1}{2}\tau\left(\left(a_{,21}a_{,1}a^{-2}\right)_{,2}\right)+\frac{5}{8}\tau\left(\left(a_{,2}a_{,1}{}^{2}a^{-3}\right)_{,2}\right)+\frac{1}{2}\tau\left(\left(a_{,22}a_{,1}a^{-2}\right)_{,1}\right)-\frac{3}{8}\tau\left(\left(a_{,2}{}^{2}a_{,1}a^{-3}\right)_{,1}\right)-\\
&\quad\frac{1}{2}\tau\left(\left(a_{,31}a_{,1}a^{-2}\right)_{,3}\right)+\frac{5}{8}\tau\left(\left(a_{,3}a_{,1}{}^{2}a^{-3}\right)_{,3}\right)+\frac{1}{2}\tau\left(\left(a_{,33}a_{,1}a^{-2}\right)_{,1}\right)-\frac{3}{8}\tau\left(\left(a_{,3}{}^{2}a_{,1}a^{-3}\right)_{,1}\right)-\\
&\quad\frac{1}{2}\tau\left(\left(a_{,41}a_{,1}a^{-2}\right)_{,4}\right)+\frac{5}{8}\tau\left(\left(a_{,4}a_{,1}{}^{2}a^{-3}\right)_{,4}\right)+\frac{1}{2}\tau\left(\left(a_{,44}a_{,1}a^{-2}\right)_{,1}\right)-\frac{3}{8}\tau\left(\left(a_{,4}{}^{2}a_{,1}a^{-3}\right)_{,1}\right)+\\
&\quad\frac{1}{4}\tau\left(\left(a_{,22}a_{,3}a^{-2}\right)_{,3}\right)-\frac{1}{4}\tau\left(\left(a_{,23}a_{,3}a^{-2}\right)_{,2}\right)+\frac{1}{4}\tau\left(\left(a_{,33}a_{,2}a^{-2}\right)_{,2}\right)-\frac{1}{4}\tau\left(\left(a_{,32}a_{,2}a^{-2}\right)_{,3}\right)+\\
&\quad\frac{1}{8}\tau\left(\left(a_{,2}a_{,3}{}^{2}a^{-3}\right)_{,2}\right)+\frac{1}{8}\tau\left(\left(a_{,3}a_{,2}{}^{2}a^{-3}\right)_{,3}\right)+\frac{1}{4}\tau\left(\left(a_{,22}a_{,4}a^{-2}\right)_{,4}\right)-\frac{1}{4}\tau\left(\left(a_{,24}a_{,4}a^{-2}\right)_{,2}\right)+\\
&\quad\frac{1}{4}\tau\left(\left(a_{,44}a_{,2}a^{-2}\right)_{,2}\right)-\frac{1}{4}\tau\left(\left(a_{,42}a_{,2}a^{-2}\right)_{,4}\right)+\frac{1}{8}\tau\left(\left(a_{,2}a_{,4}{}^{2}a^{-3}\right)_{,2}\right)+\frac{1}{8}\tau\left(\left(a_{,4}a_{,2}{}^{2}a^{-3}\right)_{,4}\right)+\\
&\quad\frac{1}{4}\tau\left(\left(a_{,33}a_{,4}a^{-2}\right)_{,4}\right)-\frac{1}{4}\tau\left(\left(a_{,34}a_{,4}a^{-2}\right)_{,3}\right)+\frac{1}{4}\tau\left(\left(a_{,44}a_{,3}a^{-2}\right)_{,3}\right)-\frac{1}{4}\tau\left(\left(a_{,43}a_{,3}a^{-2}\right)_{,4}\right)+\\
&\quad\frac{1}{8}\tau\left(\left(a_{,3}a_{,4}{}^{2}a^{-3}\right)_{,3}\right)+\frac{1}{8}\tau\left(\left(a_{,4}a_{,3}{}^{2}a^{-3}\right)_{,4}\right)+\\
&\quad\frac{1}{8}\tau\left(\left(a_{,1}{}^{3}a^{-3}\right)_{,1}\right)+\frac{1}{8}\tau\left(\left(a_{,2}{}^{3}a^{-3}\right)_{,2}\right)+\frac{1}{8}\tau\left(\left(a_{,3}{}^{3}a^{-3}\right)_{,3}\right)+\frac{1}{8}\tau\left(\left(a_{,4}{}^{3}a^{-3}\right)_{,4}\right)\\
&=0.
\end{aligned}$$

$\square$

Remark. This is a direct extension of Theorem 2.1.3 for $n=4$.

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